Distribution Algebras on p-adic Groups and Lie Algebras

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Abstract. When $F$ is a $p$-adic field, and $G = G(F)$ is the group of $F$-rational points of a connected algebraic $F$-group, the complex vector space $\mathcal{H}(G)$ of compactly supported locally constant distributions on $G$ has a natural convolution product that makes it into a $C$-algebra (without an identity) called the Hecke algebra. The Hecke algebra is a partial analogue for $p$-adic groups of the enveloping algebra of a Lie group. However, $\mathcal{H}(G)$ has drawbacks such as the lack of an identity element, and the process $G \mapsto \mathcal{H}(G)$ is not a functor. Bernstein introduced an enlargement $\mathcal{H}^\wedge(G)$ of $\mathcal{H}(G)$. The algebra $\mathcal{H}^\wedge(G)$ consists of the distributions that are left essentially compact. We show that the process $G \mapsto \mathcal{H}^\wedge(G)$ is a functor. If $\tau : G \rightarrow H$ is a morphism of $p$-adic groups, let $F(\tau) : \mathcal{H}^\wedge(G) \rightarrow \mathcal{H}^\wedge(H)$ be the morphism of $C$-algebras. We identify the kernel of $F(\tau)$ in terms of $\text{Ker}(\tau)$. In the setting of $p$-adic Lie algebras, with $\mathfrak{g}$ a reductive Lie algebra, $\mathfrak{m}$ a Levi, and $\tau : \mathfrak{g} \rightarrow \mathfrak{m}$ the natural projection, we show that $F(\tau)$ maps $G$-invariant distributions on $\mathfrak{g}$ to $N_G(\mathfrak{m})$-invariant distributions on $\mathfrak{m}$. Finally, we exhibit a natural family of $G$-invariant essentially compact distributions on $\mathfrak{g}$ associated with a $G$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ and in the case of $SL(2)$ show how certain members of the family can be moved to the group.

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