Special Values of Class Group $L$-Functions for CM Fields

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Abstract. Let $H$ be the Hilbert class field of a CM number field $K$ with maximal totally real subfield $F$ of degree $n$ over $\mathbb{Q}$. We evaluate the second term in the Taylor expansion at $s = 0$ of the Galois-equivariant $L$-function $\Theta_{S_\infty}(s)$ associated to the unramified abelian characters of $\text{Gal}(H/K)$. This is an identity in the group ring $C[\text{Gal}(H/K)]$ expressing $\Theta_{S_\infty}^{(n)}(0)$ as essentially a linear combination of logarithms of special values $\{\Psi(z_{\sigma})\}$, where $\Psi: \mathbb{H}^n \to \mathbb{R}$ is a Hilbert modular function for a congruence subgroup of $\text{SL}_2(\mathcal{O}_F)$ and $\{z_{\sigma}: \sigma \in \text{Gal}(H/K)\}$ are CM points on a universal Hilbert modular variety. We apply this result to express the relative class number $h_H/h_K$ as a rational multiple of the determinant of an $(h_K - 1) \times (h_K - 1)$ matrix of logarithms of ratios of special values $\Psi(z_{\sigma})$, thus giving rise to candidates for higher analogs of elliptic units. Finally, we obtain a product formula for $\Psi(z_{\sigma})$ in terms of exponentials of special values of $L$-functions.

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