On the Convergence of a Class of Nearly Alternating Series

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Abstract. Let $C$ be the class of convex sequences of real numbers. The quadratic irrational numbers can be partitioned into two types as follows. If $\alpha$ is of the first type and $(c_k) \in C$, then $\sum (-1)^{\lfloor k\alpha \rfloor} c_k$ converges if and only if $c_k \log k \to 0$. If $\alpha$ is of the second type and $(c_k) \in C$, then $\sum (-1)^{\lfloor k\alpha \rfloor} c_k$ converges if and only if $\sum c_k/k$ converges. An example of a quadratic irrational of the first type is $\sqrt{2}$, and an example of the second type is $\sqrt{3}$. The analysis of this problem relies heavily on the representation of $\alpha$ as a simple continued fraction and on properties of the sequences of partial sums $S(n) = \sum_{k=1}^{n} (-1)^{\lfloor k\alpha \rfloor}$ and double partial sums $T(n) = \sum_{k=1}^{n} S(k)$. 

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