Extremal Metric for the First Eigenvalue on a Klein Bottle

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Abstract. The first eigenvalue of the Laplacian on a surface can be viewed as a functional on the space of Riemannian metrics of a given area. Critical points of this functional are called extremal metrics. The only known extremal metrics are a round sphere, a standard projective plane, a Clifford torus and an equilateral torus. We construct an extremal metric on a Klein bottle. It is a metric of revolution, admitting a minimal isometric embedding into a sphere $S^4$ by the first eigenfunctions. Also, this Klein bottle is a bipolar surface for Lawson’s $\tau_{3,1}$-torus. We conjecture that an extremal metric for the first eigenvalue on a Klein bottle is unique, and hence it provides a sharp upper bound for $\lambda_1$ on a Klein bottle of a given area. We present numerical evidence and prove the first results towards this conjecture.