NILPOTENT ORBIT VARIETIES AND THE ATOMIC DECOMPOSITION OF THE $q$-KOSTKA POLYNOMIALS

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ABSTRACT. We study the coordinate rings $k[C_{n} \cap t]$ of scheme-theoretic intersections of nilpotent orbit closures with the diagonal matrices. Here $\mu'$ gives the Jordan block structure of the nilpotent matrix. de Concini and Procesi [5] proved a conjecture of Kraft [12] that these rings are isomorphic to the cohomology rings of the varieties constructed by Springer [22, 23]. The famous $q$-Kostka polynomial $K_{\mu'}(q)$ is the Hilbert series for the multiplicity of the irreducible symmetric group representation indexed by $\mu$ in the ring $k[C_{n} \cap t]$. Lascoux and Schützenberger [15, 13] gave combinatorially a decomposition of $K_{\mu'}(q)$ as a sum of “atomic” polynomials with non-negative integer coefficients, and Lascoux proposed a corresponding decomposition in the cohomology model.

Our work provides a geometric interpretation of the atomic decomposition. The Frobenius-splitting results of Mehta and van der Kallen [19] imply a direct-sum decomposition of the ideals of nilpotent orbit closures, arising from the inclusions of the corresponding sets. We carry out the restriction to the diagonal using a recent theorem of Broer [3]. This gives a direct-sum decomposition of the ideals yielding the $k[C_{n} \cap t]$, and a new proof of the atomic decomposition of the $q$-Kostka polynomials.