Canadian Mathematical Society

Sociéte mathématique du Canada

Report of the Forty-Third

Canadian Mathematical Olympiad

2011

The CMS gratefully acknowledges the support of our title sponsor:

Sun Life Financial

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The Sun Life Financial Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Sun Life Financial Canadian Open Mathematics Challenge (COMC). This year, 51 students with the highest COMC scores were invited to write the CMO. Another 150 students, next in rank, were invited to send solutions to the 10 problems of the CMO Qualifying Repêchage. The top 28 students from the Repêchage were then invited to write the CMO. A couple of additional students were invited based on other high school competition results and participation in the winter training camp.

The Society is grateful for the support from **Sun Life Financial** and the other sponsors listed on the previous page.

I am very grateful to the CMO Committee members for submitting the problems, reviewing the test and marking the solutions: Andrew Adler, Jason Bell, Julia Gordon, Robert Morewood, Zinovy Reichstein, Naoki Sato, Jozsef Solymosi and Adrian Tang. Thanks also go to David Arthur, Tuan Le and Jon Schneider for submitting problems for the CMO and to Joseph Khoury for translating the problems into French. The CMO Qualifying Repêchage was organized by Ian VanderBurgh at the University of Waterloo. Finally, I would like to thank Laura Alyea and the CMS Executive Director Johan Rudnick for the hard work done at the CMS headquarters.

Kalle Karu, Chair
Canadian Mathematical Olympiad Committee

The 43rd Canadian Mathematical Olympiad was written on Wednesday, March 23, 2011. A total of 81 sets of solutions were received. Of these, 73 were from Canadian schools; the remaining sets were from Canadian students studying abroad. All were eligible for CMO official prizes. Six Canadian provinces were represented, with the number of contestants as follows:

AB (6) BC (18) NB (1) ON (47) QC (1) SK (2)

The 2011 CMO consisted of five questions, each marked out of seven. The maximum score obtained by the winner was 33 marks. The contestants were grouped into four divisions according to their scores as follows:

Division	Range of Scores	# of Students
I	23-33	9
II	16-22	15
III	12-15	19
IV	1-14	38

FIRST PRIZE – Sun Life Financial Cup - $2000

Mariya Sardarli
Strathcona High School, Edmonton, AB

SECOND PRIZE - $1500

Calvin Deng
William G. Enloe High School, Raleigh, NC

THIRD PRIZE - $1000

Alex Song
Detroit Country Day School, Beverly Hills, MI

HONOURABLE MENTIONS - $500

Matthew Brennan
Upper Canada College, Toronto, ON

Yuzhou Chen
Sir John A. Macdonald Collegiate Institute, Agincourt, ON

Yi Liu
York Mills Collegiate Institute, Toronto, ON

James Rickards
Colonel By Secondary School, Gloucester, ON

Hunter Spink
Western Canada High School, Calgary, AB

Susan Sun
West Vancouver Secondary School, West Vancouver, BC

Division I 23-33

Mariya Sardarli	Strathcona H.S.	AB
Calvin Deng	William G. Enloe H.S.	NC
Alex Song	Detroit Country Day School	MI
Matthew Brennan	Upper Canada College	ON
Yuzhou Chen	Sir John A. Macdonald C.I.	ON
Yi Liu	York Mills C.I.	ON
James Rickards	Colonel By S.S.	ON
Hunter Spink	Western Canada H.S.	ON
Susan Sun	West Vancouver S.S.	BC

Division 2 16-22

Yaroslav Babich	Sir Winston Churchill H.S.	AB
Weilian Chu	Old Scona Academic	AB
Yun Jia (Melody) Guan	University of Toronto Schools	ON
Heinrich Jiang	Honourable Vincent Massey S.S.	ON
Dong Won Kang	North Toronto C.I.	ON
Leo Lai	Sir Winston Churchill S.S,	BC
Daniel Spivak	Bayview Secondary School	ON
Zihao Wang	Lord Byng S.S.	BC
Jay Young Woo	London Central S.S.	ON
Yu Wu	Agincourt C.I.	ON
Allen Yang	Cary Academy	NC
Steven Yu	Pinetree S.S.	BC
Joe Zeng	Don Mills C.I.	ON
Cyril Xi Yao Zhang	Don Mills C.I.	ON
Kevin Zhou	Woburn C.I.	ON

Division 3 12-15

Bardia Beigi	West Vancouver Secondary School	BC
Wonjohn Choi	St. Francis Xavier S.S.	ON
Liqing Ding	Branksome Hall	ON
Vahid Fazel-Rezai	Red River H.S.	ND
Alexandru Gatea	Waterloo C.I.	ON
Tian Lan	Northern S.S.	ON
Kevin Lau	Richmond Hill H.S.	ON
Kevin Michael Li	A&M Consolidated H.S.	TX
Xu Lin	H.B. Beal Secondary	ON
David Si Qi Liu	Honourable Vincent Massey S.S.	ON
Kevin Luo	Eric Hamber Secondary	BC
Anupa Murali	Bishop Brady High School	NH
Yingjie Qian	Bulkley Valley Christian School	BC

Henry Heng Tang	Bayview Secondary School	ON
Chao Wang	Sir John A. Macdonald C.I.	ON
Kaiyu Wu	Meadowville S.S.	ON
Eric Zhan	University of Toronto Schools	ON
Tianchen Zhao	Point Grey Secondary	BC
Kaiven Zhou	Strathcona H.S.	AB

Division 4 1-14

Sifan Bi	Sir John A. Macdonald S.S.	ON
Zhiming Chen	Evan Hardy Collegiate	SK
Da Qi Chen	Marianopolis College	QC
Gregory Chen	Thornhill S.S.	ON
Daniel Chong	A.B. Lucas S.S.	ON
Anqi Dong	Walter Murray C.I.	SK
Rongxin Du	Olympiads School	ON
Jimmy Fang	St. Robert Catholic H.S.	ON
Xin Yue (Hino) Feng	A.Y. Jackson S.S.	ON
Lanxin (Fiona) Gao	A.Y. Jackson S.S.	ON
Jihyen Ha	Kennebecasis Valley H.S.	NB
Tim He	Henry Wise Wood H.S.	AB
Daniel Hu	Honourable Vincent Massey S.S.	ON
Billy Janitsch	Earl Haig S.S.	ON
Jongsoo Lee	Earl Haig S.S.	ON
Ursula Anne Lim	Burnaby North Secondary	BC
Ken Lin	Don Mills C.I.	ON
Boris Lin	Elgin Park Secondary	BC
Rick Lu	Claremont Secondary	BC
Richard Luo	A.R. MacNeill S.S.	BC
Matthew Ng	St. Francis Xavier S.S.	ON
Jee Young (Gayle) Oh	Lord Byng Secondary	BC
Soohyun Park	University of Toronto Schools	ON
Ritvik Ramkumar	Glenforest S.S.	ON
Samer Seraj	John Fraser S.S.	ON
Zheng Wang	Hugh Boyd S.S.	BC
Jesse Kent Wang	Lisgar C.I.	ON
Tony Wu	Dr. Norman Bethune C.I.	ON
Yongyi Wu	Lester B. Pearson College	BC
Xiaosong Yin	Magee Secondary	BC
Fan Yin	Honourable Vincent Massey S.S.	ON
Simon Younan	St. Francis Xavier S.S.	ON
Cheng Zeng	The Woodlands School	ON
Xiaoying (Gillian) Zhang	A.Y. Jackson S.S.	ON
Gavin Zhang	Sir Winston Churchill S.S.	BC
Bill Zhao	A.Y. Jackson S.S.	ON
Stephen Zhou	Lord Byng S.S.	BC
Jia Lin Zhu	Unionville H.S.	ON

The Grader's Report

The 2011 CMO was marked two weeks after the competition date by Andrew Adler, Julia Gordon, Kalle Karu, Robert Morewood and Zinovy Reichstein. All 81 papers were marked once, and then the top scoring papers were marked again

The problems this year were slightly easier than in the past. However, no student received a perfect score, with the top three scores being 33, 30 and 29 out of 35. The median score was 13. The table below lists the number of students receiving a given number of points for each problem. The last line (indicated by "--") shows the number of students who did not attempt the problem.

Score	Problem #1	Problem #2	Problem #3	Problem #4	Problem #5
7	24	45	1	17	1
6	8	1	1	0	1
5	10	5	1	0	2
4	7	3	6	0	1
3	6	3	1	3	3
2	4	2	18	2	1
1	6	14	28	0	3
0	15	4	15	27	24
--	1	4	10	32	45

Problem #1. The official solution uses a clever divisibility condition. Recognizing this leads to a very short solution. Another approach is to use the fact that quotient of any two numbers from the set is at most 9 and then try to rule out each such nontrivial quotient. This seemingly simpler start leads to many cases to be considered, and hence resulted in many part marks.

Problem #2. The Euclidean geometry problem was the easiest problem of the test, with more than half the students getting maximum points. There were many approaches, such as chasing angles, using the sine or cosine laws and other theorems from geometry. One has to use the fact that the quadrilateral is cyclic, but then almost all approaches lead to correct solutions. A typical strategy was to prove that the two angle bisectors intersect at a right angle and deduce the result from there.

Problem #3. This problem and Problem #5 were the hardest problems of the test. There were very few complete or near complete solutions. A number of students received one or two part marks because they found the correct bound and proved some special cases. Realizing that the Cauchy-Schwartz inequality is important gave a few more part marks.

Problem #4. The fourth problem turned out to be quite simple. The pigeonhole principle is well-known and its application here is straightforward. Everybody who got so far seemed to know that 2011 is a prime number. There was a large number of students who got zero marks or who did not attempt the problem, probably because it came so late in the test.

Problem #5. The last problem was no doubt the hardest one in the test, with 69 students getting no points at all. It should be mentioned that the top three students did give a (near) complete solution to this problem.

Appendix

43rd Canadian Mathematical Olympiad 2011

Problems and Solutions

CANADIAN MATHEMATICAL OLYMPIAD 2011
PROBLEMS AND SOLUTIONS

(1) Consider 70-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n ten times (and 8, 9, and 0 do not appear). Show that no number of this form can divide another number of this form.

Solution. Assume the contrary: there exist a and b of the prescribed form, such that $b \geq a$ and a divides b. Then a divides $b - a$.

Claim: a is not divisible by 3 but $b - a$ is divisible by 9. Indeed, the sum of the digits is $10(1 + \cdots + 7) = 280$, for both a and b. [Here one needs to know or prove that an integer n is equivalent of the sum of its digits modulo 3 and modulo 9.]

We conclude that $b - a$ is divisible by $9a$. But this is impossible, since $9a$ has 71 digits and b has only 70 digits, so $9a > b > b - a$. □

(2) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

Solution. Since $ABCD$ is cyclic, $\Delta XAC \sim \Delta XDB$ and $\Delta YAC \sim \Delta YBD$. Therefore,

$$\frac{XA}{XD} = \frac{XC}{XB} = \frac{AC}{DB} = \frac{YA}{YB} = \frac{YC}{YD}.$$

Let s be this ratio. Therefore, by the angle bisector theorem,

$$\frac{AE}{ED} = \frac{XA}{XD} = \frac{XC}{XB} = \frac{CF}{FB} = s,$$

and

$$\frac{AG}{GB} = \frac{YA}{YB} = \frac{YC}{YD} = \frac{CH}{HD} = s.$$

Hence, $\frac{AG}{GB} = \frac{CF}{FB}$ and $\frac{AE}{ED} = \frac{DH}{HC}$. Therefore, $EH || AC || GF$ and $EG || DB || HF$. Hence, $EGFH$ is a parallelogram. □

(3) Amy has divided a square up into finitely many white and red rectangles, each with sides parallel to the sides of the square. Within each white rectangle, she writes down its width divided by its height. Within each red rectangle, she writes down its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total area of the white rectangles equals the total area of the red rectangles, what is the smallest possible value of x?

Solution. Let a_i and b_i denote the width and height of each white rectangle, and let c_i and d_i denote the width and height of each red rectangle. Also, let L denote the side length of the original square.

Lemma: Either $\sum a_i \geq L$ or $\sum d_i \geq L$.

Proof of lemma: Suppose there exists a horizontal line across the square that is covered entirely with white rectangles. Then, the total width of these rectangles is at least L, and the claim is proven. Otherwise, there is a red rectangle intersecting every horizontal line, and hence the total height of these rectangles is at least L. □

Now, let us assume without loss of generality that $\sum a_i \geq L$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\left(\sum \frac{a_i}{b_i}\right) \cdot \left(\sum a_i b_i\right) &\geq \left(\sum a_i\right)^2 \\ &\geq L^2.\end{aligned}$$

But we know $\sum a_i b_i = \frac{L^2}{2}$, so it follows that $\sum \frac{a_i}{b_i} \geq 2$. Furthermore, each $c_i \leq L$, so

$$\sum \frac{d_i}{c_i} \geq \frac{1}{L^2} \cdot \sum c_i d_i = \frac{1}{2}.$$

Therefore, x is at least 2.5. Conversely, $x = 2.5$ can be achieved by making the top half of the square one colour, and the bottom half the other colour. □

(4) Show that there exists a positive integer N such that for all integers $a > N$, there exists a contiguous substring of the decimal expansion of a that is divisible by 2011. (For instance, if $a = 153204$, then 15, 532, and 0 are all contiguous substrings of a. Note that 0 is divisible by 2011.)

Solution. We claim that if the decimal expansion of a has at least 2012 digits, then a contains the required substring. Let the decimal expansion of a be $a_k a_{k-1} \dots a_0$. For $i = 0, \dots, 2011$, Let b_i be the number with decimal expansion $a_i a_{i-1} \dots a_0$. Then by pidgenhole principle, $b_i \equiv b_j \mod 2011$ for some $i < j \leq 2011$. It follows that 2011 divides $b_j - b_i = c \cdot 10^i$. Here c is the substring $a_j \dots a_{i+1}$. Since 2011 and 10 are relatively prime, it follows that 2011 divides c. □

(5) Let d be a positive integer. Show that for every integer S, there exists an integer $n > 0$ and a sequence $\epsilon_1, \epsilon_2, \dots, \epsilon_n$, where for any k, $\epsilon_k = 1$ or $\epsilon_k = -1$, such that

$$S = \epsilon_1(1+d)^2 + \epsilon_2(1+2d)^2 + \epsilon_3(1+3d)^2 + \cdots + \epsilon_n(1+nd)^2.$$

Solution. Let $U_k = (1+kd)^2$. We calculate $U_{k+3} - U_{k+2} - U_{k+1} + U_k$. This turns out to be $4d^2$, a constant. Changing signs, we obtain the sum $-4d^2$.

Thus if we have found an expression for a certain number S_0 as a sum of the desired type, we can obtain an expression of the desired type for $S_0 + (4d^2)q$, for any integer q.

It remains to show that for any S, there exists an integer S' such that $S' \equiv S \pmod{4d^2}$ and S' can be expressed in the desired form. Look at the sum

$$(1+d)^2 + (1+2d)^2 + \cdots + (1+Nd)^2,$$

where N is "large." We can at will choose N so that the sum is odd, or so that the sum is even.

By changing the sign in front of $(1+kd)^2$ to a minus sign, we decrease the sum by $2(1+kd)^2$. In particular, if $k \equiv 0 \pmod{2d}$, we decrease the sum by 2 (modulo $4d^2$). So

If N is large enough, there are many $k < N$ such that k is a multiple of $2d$. By switching the sign in front of r of these, we change ("downward") the congruence class modulo $4d^2$ by $2r$. By choosing N so that the original sum is odd, and choosing suitable $r < 2d^2$, we can obtain numbers congruent to all odd numbers modulo $4d^2$. By choosing N so that the original sum is even, we can obtain numbers congruent to all even numbers modulo $4d^2$. This completes the proof. □