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The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Canadian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Canadian Open Mathematical Challenge (COMC). This year, the top 67 COMC scores were invited outright to write the CMO; all but seven accepted. Approximately 200 others, next in rank, were invited to send solutions to a Repêchage set of ten problems posted online within a week to the University of Waterloo. Thirty-five students were invited on the basis of this, of which 31 accepted. I am grateful to Ian VanderBurgh for setting this up and assembling a team of markers, consisting of Ed Anderson, Lloyd Auckland, Ed Barbeau, Enzo Carli, Eddie Cheung, Rad de Peiza, Larry Rice, Jim Schurter, Ian VanderBurgh and Kyle Willick, to go through the 126 scripts received.

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I am very grateful to the CMO Committee members who submitted problems to be considered for the 2008 competition: Ed Doolittle, Chris Fisher, Valeria Pandelieva, Naoki Sato, Adrian Tang and Jacob Tsimerman. The papers were marked by Ed Barbeau, Man-Duen Choi, Felix Recio and Lindsey Shorser. Thanks go to Tom Griffiths of London, ON and Kalle Karu of the University of British Columbia for validating the paper and to Joseph Khoury for translating the paper and the solutions into French. I am indebted for the hard work done at CMS headquarters by Susan Latreille and the Executive Director, Graham Wright, whose commitment and zeal is a vital ingredient of the success of the CMO.

Ed Barbeau, Chair
Canadian Mathematical Olympiad Committee
The 40th (2008) Canadian Mathematical Olympiad was written on Wednesday, March 26, 2008. A total of 96 students from 57 schools (52 in Canada, 3 in the US and 2 abroad) wrote the paper. Seven Canadian provinces were represented, with the number of contestants as follows:

- BC (20)
- AB (5)
- SK (3)
- MB (3)
- ON (49)
- QC (3)
- NB (1).

The 2008 CMO consisted of five questions, each marked out of 7. The maximum score attained by a student was 28. The official contestants were grouped into four divisions according to their scores as follows:

<table>
<thead>
<tr>
<th>Division</th>
<th>Range of Scores</th>
<th>No. of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>21 - 28</td>
<td>8</td>
</tr>
<tr>
<td>II</td>
<td>13 - 19</td>
<td>17</td>
</tr>
<tr>
<td>III</td>
<td>7 - 12</td>
<td>29</td>
</tr>
<tr>
<td>IV</td>
<td>0 - 6</td>
<td>42</td>
</tr>
</tbody>
</table>

The following tables give the scores obtained on the COMC along with the corresponding scores obtained by candidates on the CMO. Students obtaining scores 72-80 qualified directly to write the CMO; students with scores 65-71 qualified through the Repêchage.

80 (26, 24, 15)  
79 (21, 19, 15)  
78 (23, 12, 10)  
77 (28, 25, 18)  
76 (12, 11, 11, 7, 6)  
75 (15, 8, 5, 3)  
74 (17, 17, 14, 14, 10, 6, 6, 3, 3, 3, 1)  
73 (25, 21, 9, 9, 8, 7, 2, 2, 1, 0)  
72 (15, 15, 11, 10, 9, 7, 5, 5, 4, 4, 2, 2, 1)  
71 (19, 16, 11)  
70 (4, 1)  
69 (17, 4)  
68 (19)  
67 (13, 9, 8, 5, 1)  
66 (11, 11, 10, 9, 9, 7, 6, 4, 1)  
65 (10, 3, 3, 0)
FIRST PRIZE — Sun Life Financial Cup — $2000

Chen Sun

SECOND PRIZE — $1500

Jonathan Schneider
University of Toronto Schools, Toronto, Ontario

THIRD PRIZE — $1000

Yan Li
Dr. Norman Bethune Collegiate Institute, Toronto, Ontario

HONOURABLE MENTIONS — $500

Dimitri Dziabenko
Don Mills Collegiate Institute
Toronto, ON

Neil Gurram
ICAE
Troy, MI

Danny Shi
Sir Winston Churchill High School
Calgary, AB

Jarno Sun
Western Canada High School
Calgary, AB

Tianyao Zhang
Sir John A. Macdonald Collegiate Institute
Toronto, ON
Report - Fortieth Canadian Mathematical Olympiad 2008

Division 1
21-28

Chen Sun
Jonathan Schneider
Yan Li
Dimitri Dziabenko
Neil Gurram
Danny Shi
Jarno Sun
Tianyao Zhang
A.B. Lucas S.S.
UTS
Dr. Norman Bethune C.I.
Don Mills C.I.
ICAE
Sir Winston Churchill H.S.
Western Canada H.S.
Sir John A. Macdonald C.I.
ON
ON
ON
ON
MI
AB
AB
ON

Xiao Xu
Meng Ye
Pei Jun Zhao
Vincent Zhou
Zimu Zhu
Jonathan Zung
Georges Vanier S.S.
Marianopolis College
London Central S.S.
Dr. Norman Bethune C.I.
Richmond Hill H.S.
University of Toronto Schools
ON
QC
ON
ON
ON
ON

Division 2
13-19

Mohammad Babadi
Frank Ban
Yuhan Chen
Robin Cheng
Bo Cheng Cui
Tony Feng
David Field
Yuting Huang
Joe Kileel
Zhiqiang Liu
Jingyuan Mo
Alexander Remorov
Jojun Wang
Weinan Peter Wen
Anqi Zhang
Linda Zhang
Jonathan Zhou
Thornlea S.S.
Vincent Massey S.S.
Sir Winston Churchill C.V.I.
Pinetree S.S.
West Vancouver S.S.
Phillips Academy
Phillips Academy
Johnston Heights S.S.
Fredericton H.S.
Don Mills C.I.
St. George’s School
William Lyon Mackenzie King C.I.
Vincent Massey S.S.
Vincent Massey S.S.
Western Canada H.S.
Burnaby North S.S.
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Shek Wah Chan
Jerry Chen
Lingjun Chen
Chengcheng Gui
Adam Halski
Ding Henry Hao
Matthew Harrisontraino
Kevin He
Emily Wei-En Hsu
Yihuan Peter Huang
Navid Javadi
Chen Jiang
Yangzi Jiang
Hee Woo Jun
Randy Li
Alex Liang
William Lin
Chieh Ming Liu
Eric Liu
Xun Chao Max Liu
David Ma
Susanne Michelle Morill
Ryan Peng
Zhe Qu
Calvin Seo
Yeongseok Suh
Hao Sun
Tanya Tang
Russell Vanderhout
Kedi Wang
Richard Wang
Susan Wang
Wen Wang
Jun Wen
Carrie Xing
Vick Yao
Victor Zhang
Yunfan Zhang
Dabo Zhao
Steven Zhu
Yang Zhu
St. Paul’s Co-ed. College
Moscrop S.S.
Don Mills C.I.
St. John’s-Ravenscourt School
Kuwait English School
Albert Campbell C.I.
Marc Garneau C.I.
Sir Winston Churchill S.S.
Branksome Hall School
Kingston C.V.I.
Earl Haig S.S.
Central Technical School
Waterloo C.I.
Pinetree S.S.
Phillips Academy
Dr. Norman Bethune C.I.
Albert Campbell C.I.
Fraser Heights S.S.
Sir Winston Churchill S.S.
Port Moody S.S.
Marianopolis College
ICAE
St. Mary’s Academy
Centennial Collegiate
Sir Allan MacNab S.S.
St. Andrew’s College
York Mills C.I.
Centennial Collegiate
Sir Winston Churchill S.S.
Fraser Heights S.S.
Fort Richmond Collegiate
Sir Winston Churchill S.S.
Burnaby Central S.S.
Western Canada H.S.
London International Academy
Marc Garneau C.I.
Vincent Massey S.S.
Marc Garneau C.I.
Phillips Academy
White Oaks S.S.
Sir Winston Churchill S.S.
Albert Campbell C.I.
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Division 3
7-12

Golam Tahri’f Bappi
Shalev Ben David
Ram Bhaskar
Philip Chen
Weiliang Chen
Andrew Dhawan
Henry Fung
Fang Guo
Tony Han
Fan Jiang
Heinrich Jiang
Kwon Yong Jin
Eric Gwangseung Kim
Jung Hun Koh
Nikita Lvov
Anupa Murali
Bill Pang
Owen Zhu Ren
Mariya Sardarli
Alex Song
Julian Sun
Ning Tang
Ming Jing Wong
Waterloo C.I.
Waterloo C.I.
ICAE
Glenforest S.S.
The Woodlands School
Glenforest S.S.
Richmond Hill H.S.
Jarvis C.I.
Albert Campbell C.I.
Vincent Massey S.S.
Phillips Academy
Prince of Wales S.S.
Phillips Academy
Marianopolis College
Derryfield School
Sir Winston Churchill S.S.
Sir Winston Churchill S.S.
London International Academy
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Shek Wah Chan
Jerry Chen
Lingjun Chen
Chengcheng Gui
Adam Halski
Ding Henry Hao
Matthew Harrisontraino
Kevin He
Emily Wei-En Hsu
Yihuan Peter Huang
Navid Javadi
Chen Jiang
Yangzi Jiang
Hee Woo Jun
Randy Li
Alex Liang
William Lin
Chieh Ming Liu
Eric Liu
Xun Chao Max Liu
David Ma
Susanne Michelle Morill
Ryan Peng
Zhe Qu
Calvin Seo
Yeongseok Suh
Hao Sun
Tanya Tang
Russell Vanderhout
Kedi Wang
Richard Wang
Susan Wang
Wen Wang
Jun Wen
Carrie Xing
Vick Yao
Victor Zhang
Yunfan Zhang
Dabo Zhao
Steven Zhu
Yang Zhu
St. Paul’s Co-ed. College
Moscrop S.S.
Don Mills C.I.
St. John’s-Ravenscourt School
Kuwait English School
Albert Campbell C.I.
Marc Garneau C.I.
Sir Winston Churchill S.S.
Branksome Hall School
Kingston C.V.I.
Earl Haig S.S.
Central Technical School
Waterloo C.I.
Pinetree S.S.
Phillips Academy
Dr. Norman Bethune C.I.
Albert Campbell C.I.
Fraser Heights S.S.
Sir Winston Churchill S.S.
Port Moody S.S.
Marianopolis College
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St. Andrew’s College
York Mills C.I.
Centennial Collegiate
Sir Winston Churchill S.S.
Fraser Heights S.S.
Fort Richmond Collegiate
Sir Winston Churchill S.S.
Burnaby Central S.S.
Western Canada H.S.
London International Academy
Marc Garneau C.I.
Vincent Massey S.S.
Marc Garneau C.I.
Phillips Academy
White Oaks S.S.
Sir Winston Churchill S.S.
Albert Campbell C.I.
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4
1. \(ABCD\) is a convex quadrilateral for which \(AB\) is the longest side. Points \(M\) and \(N\) are located on sides \(AB\) and \(BC\) respectively, so that each of the segments \(AN\) and \(CM\) divides the quadrilateral into two parts of equal area. Prove that the segment \(MN\) bisects the diagonal \(BD\).

2. Determine all functions \(f\) defined on the set of rational numbers that take rational values for which

\[
f(2f(x) + f(y)) = 2x + y,
\]

for each \(x\) and \(y\).

3. Let \(a, b, c\) be positive real numbers for which \(a + b + c = 1\). Prove that

\[
\frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2}.
\]

4. Determine all functions \(f\) defined on the natural numbers that take values among the natural numbers for which

\[
(f(n))^p \equiv n \mod f(p)
\]

for all \(n \in \mathbb{N}\) and all prime numbers \(p\).

5. A self-avoiding rook walk on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook’s path is non-self-intersecting.

Let \(R(m, n)\) be the number of self-avoiding rook walks on an \(m \times n\) (\(m\) rows, \(n\) columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, \(R(m, 1) = 1\) for all natural numbers \(m\); \(R(2, 2) = 2\); \(R(3, 2) = 4\); \(R(3, 3) = 11\). Find a formula for \(R(3, n)\) for each natural number \(n\).
40th Canadian Mathematical Olympiad

Wednesday, March 26, 2008

Solutions - CMO 2008

1. \(ABCD\) is a convex quadrilateral in which \(AB\) is the longest side. Points \(M\) and \(N\) are located on sides \(AB\) and \(BC\) respectively, so that each of the segments \(AN\) and \(CM\) divides the quadrilateral into two parts of equal area. Prove that the segment \(MN\) bisects the diagonal \(BD\).

   Solution. Since \([MADC] = \frac{1}{2}[ABCD] = [NADC]\), it follows that \([ANC] = [AMC]\), so that \(MN\parallel AC\). Let \(m\) be a line through \(D\) parallel to \(AC\) and \(MN\) and let \(BA\) produced meet \(m\) at \(P\) and \(BC\) produced meet \(m\) at \(Q\). Then

   \[
   [MPC] = [MAC] + [CAP] = [MAC] + [CAD] = [MADC] = [BMC]
   \]

   whence \(BM = MP\). Similarly \(BN = NQ\), so that \(MN\) is a midline of triangle \(BPQ\) and must bisect \(BD\).

2. Determine all functions \(f\) defined on the set of rationals that take rational values for which

   \[
   f(2f(x) + f(y)) = 2x + y
   \]

   for each \(x\) and \(y\).

   Solution 1. The only solutions are \(f(x) = x\) for all rational \(x\) and \(f(x) = -x\) for all rational \(x\). Both of these readily check out.

   Setting \(y = x\) yields \(f(3f(x)) = 3x\) for all rational \(x\). Now replacing \(x\) by \(3f(x)\), we find that

   \[
   f(9x) = f(3f(3f(x)) = 3[3f(x)] = 9f(x),
   \]

   for all rational \(x\). Setting \(x = 0\) yields \(f(0) = 9f(0)\), whence \(f(0) = 0\).

   Setting \(x = 0\) in the given functional equation yields \(f(f(y)) = y\) for all rational \(y\). Thus \(f\) is one-one onto. Applying \(f\) to the functional equation yields that

   \[
   2f(x) + f(y) = f(2x + y)
   \]

   for every rational pair \((x, y)\).

   Setting \(y = 0\) in the functional equation yields \(f(2f(x)) = 2x\), whence \(2f(x) = f(2x)\). Therefore \(f(2x) + f(y) = f(2x + y)\) for each rational pair \((x, y)\), so that

   \[
   f(u + v) = f(u) + f(v)
   \]

   for each rational pair \((u, v)\).
Since $0 = f(0) = f(-1) + f(1)$, $f(-1) + f(1) = -f(1)$. By induction, it can be established that for each integer $n$ and rational $x$, $f(nx) = nf(x)$. If $k = f(1)$, we can establish from this that $f(n) = nk$, $f(1/n) = k/n$ and $f(m/n) = mk/n$ for each integer pair $(m, n)$. Thus $f(x) = kx$ for all rational $x$. Since $f(f(x)) = x$, we must have $k^2 = 1$. Hence $f(x) = x$ or $f(x) = -x$. These check out.

Solution 2. In the functional equation, let
\[ x = y = 2f(z) + f(w) \]
to obtain $f(x) = f(y) = 2z + w$ and
\[ f(6z + 3w) = 6f(z) + 3f(w) \]
for all rational pairs $(z, w)$. Set $(z, w) = (0, 0)$ to obtain $f(0) = 0$, $w = 0$ to obtain $f(6z) = 6f(z)$ and $z = 0$ to obtain $f(3w) = 3f(w)$ for all rationals $z$ and $w$. Hence $f(6z + 3w) = f(6z) + f(3w)$. Replacing $(6z, 3w)$ by $(u, v)$ yields
\[ f(u + v) = f(u) + f(v) \]
for all rational pairs $(u, v)$. Hence $f(x) = kx$ where $k = f(1)$ for all rational $x$. Substitution of this into the functional equation with $(x, y) = (1, 1)$ leads to $3 = f(3f(1)) = f(3k) = 3k^2$, so that $k = \pm 1$. It can be checked that both $f(x) \equiv 1$ and $f(x) \equiv -1$ satisfy the equation.

Acknowledgment. The first solution is due to Man-Duen Choi and the second to Ed Doolittle.

3. Let $a, b, c$ be positive real numbers for which $a + b + c = 1$. Prove that
\[ \frac{a - bc}{a + bc} + \frac{b - ca}{b + ca} + \frac{c - ab}{c + ab} \leq \frac{3}{2}. \]

Solution 1. Note that
\[ 1 - \frac{a - bc}{a + bc} = \frac{2bc}{1 - b - c + bc} = \frac{2bc}{(1 - b)(1 - c)}. \]
The inequality is equivalent to
\[ \frac{2bc}{(1 - b)(1 - c)} + \frac{2ca}{(1 - c)(1 - a)} + \frac{2ab}{(1 - a)(1 - b)} \geq \frac{3}{2}. \]
Manipulation yields the equivalent
\[ 4(bc + ca + ab - 3abc) \geq 3(bc + ca + ab + 1 - a - b - c - abc). \]
This simplifies to $ab + bc + ca \geq 9abc$ or
\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9. \]
This is a consequence of the harmonic-arithmetic means inequality.

Solution 2. Observe that
\[ a + bc = a(a + b + c) + bc = (a + b)(a + c) \]
and that $a + b = 1 - c$, with analogous relations for other permutations of the variables. Then
\[ (b + c)(c + a)(a + b) = (1 - a)(1 - b)(1 - c) = (ab + bc + ca) - abc. \]
Putting the left side of the desired inequality over a common denominator, we find that it is equal to
\[(a - bc)(1 - a) + (b - ac)(1 - b) + (c - ab)(1 - c)\]
\[= \frac{(a + b + c) - (a^2 + b^2 + c^2) - (bc + ca + ab) + 3abc}{(b + c)(c + a)(a + b)}\]
\[= 1 - \frac{(a + b + c)^2 + (bc + ca + ab) + 3abc}{(ab + bc + ca) - abc}\]
\[= \frac{(bc + ca + ab) + 3abc}{(bc + ca + ab) - abc} - 4abc\]
\[= 1 + \frac{1}{(b + c)(c + a)} .\]

Using the arithmetic-geometric means inequality, we obtain that
\[(a + b)(b + c)(c + a) = (a^2b + b^2c + c^2a) + (ab2 + be2 + ca2) + 2abc\]
\[\geq 3abc + 3abc + 2abc = 8abc ,\]
whence \(4abc/[a + b + c](c + a)] \leq \frac{1}{2}. The desired result follows. Equality occurs exactly when \(a = b = c = \frac{1}{2}\).

4. Find all functions \(f\) defined on the natural numbers that take values among the natural numbers for which
\[(f(n))^p \equiv n \mod f(p)\]
for all \(n \in \mathbb{N}\) and all prime numbers \(p\).

Solution. The substitution \(n = p\), a prime, yields \(p \equiv (f(p))^p \equiv 0 \mod f(p)\), so that \(p\) is divisible by \(f(p)\). Hence, for each prime \(p\), \(f(p) = 1\) or \(f(p) = p\).

Let \(S = \{p : p\) is prime and \(f(p) = p\}\). If \(S\) is infinite, then \(f(n)^p \equiv n \pmod{f(p)}\) for infinitely many primes \(p\). By the little Fermat theorem, \(n \equiv f(n)^p \equiv f(n)\), so that \(f(n) - n\) is a multiple of \(p\) for infinitely many primes \(p\). This can happen only if \(f(n) = n\) for all values of \(n\), and it can be verified that this is a solution.

If \(S\) is empty, then \(f(p) = 1\) for all primes \(p\), and any function satisfying this condition is a solution.

Now suppose that \(S\) is finite and non-empty. Let \(q\) be the largest prime in \(S\). Suppose, if possible, that \(q \geq 3\). Therefore, for any prime \(p\) exceeding \(q\), \(p \equiv 1 \mod q\). However, this is not true. Let \(Q\) be the product of all the odd primes up to \(q\). Then \(Q + 2\) must have a prime factor exceeding \(q\) and at least one of them must be incongruent to \(1 \mod q\). (An alternative argument notes that Bertrand’s postulate can turn up a prime \(p\) between \(q\) and \(2q\) which fails to satisfy \(p \equiv 1 \mod q\).)

The only remaining case is that \(S = \{2\}\). Then \(f(2) = 2\) and \(f(p) = 1\) for every odd prime \(p\). Since \(f(n)^2 \equiv n \pmod{2}\), \(f(n)\) and \(n\) must have the same parity. Conversely, any function \(f\) for which \(f(n) \equiv n \pmod{2}\) for all \(n\), \(f(2) = 2\) and \(f(p) = 1\) for all odd primes \(p\) satisfies the condition.

Therefore the only solutions are
- \(f(n) = n\) for all \(n \in \mathbb{N}\);
- any function \(f\) with \(f(p) = 1\) for all primes \(p\);
- any function for which \(f(2) = 2\), \(f(p) = 1\) for primes \(p\) exceeding 2 and \(f(n)\) and \(n\) have the same parity.

5. A self-avoiding rook walk on a chessboard (a rectangular grid of squares) is a path traced by a sequence of rooks moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rook’s path is non-self-intersecting.

Let \(R(m, n)\) be the number of self-avoiding rook walks on an \(m \times n\) (\(m\) rows, \(n\) columns) chessboard which begin at the lower-left corner and end at the upper-left corner. For example, \(R(m, 1) = 1\) for all
natural numbers \( m \): \( R(2, 2) = 2; R(3, 2) = 4; R(3, 3) = 11 \). Find a formula for \( R(3, n) \) for each natural number \( n \).

Solution 1. Let \( r_n = R(3, n) \). It can be checked directly that \( r_1 = 1 \) and \( r_2 = 4 \). Let \( 1 \leq i \leq 3 \) and \( 1 \leq j \); let \((i, j)\) denote the cell in the \( i \)th row from the bottom and the \( j \)th column from the left, so that the paths in question go from \((1, 1)\) to \((3, 1)\).

Suppose that \( n \geq 3 \). The rook walks fall into exactly one of the following six categories:

1. One walk given by \((1, 1) \to (2, 1) \to (3, 1)\).
2. Walks that avoid the cell \((2, 1)\): Any such walk must start with \((1, 1) \to (1, 2)\) and finish with \((3, 2) \to (3, 1)\); there are \( r_{n-1} \) such walks.
3. Walks that begin with \((1, 1) \to (2, 1) \to (2, 2)\) and never return to the first row: Such walks enter the third row from \((2, k)\) for some \( k \) with \( 2 \leq k \leq n \) and then go along the third row leftwards to \((3, 1)\); there are \( n - 1 \) such walks.
4. Walks that begin with \((1, 1) \to (2, 1) \to \cdots \to (2, k) \to (1, k) \to (1, k + 1)\) and end with \((3, k + 1) \to (3, k) \to (3, k - 1) \to \cdots \to (3, 2) \to (3, 1)\) for some \( k \) with \( 2 \leq k \leq n - 1 \); there are \( r_{n-2} + r_{n-3} + \cdots + r_1 \) such walks.
5. Walks that are the horizontal reflected images of walks in (3) that begin with \((1, 1) \to (2, 1)\) and never enter the third row until the final cell; there are \( n - 1 \) such walks.
6. Walks that are horizontal reflected images of walks in (5); there are \( r_{n-2} + r_{n-3} + \cdots + r_1 \) such walks.

Thus, \( r_3 = 1 + r_2 + 2(2 + r_1) = 11 \) and, for \( n \geq 3 \),
\[
  r_n = r_{n-1} + 2[(n - 1) + r_{n-2} + r_{n-3} + \cdots + r_1]
  = 2n - 1 + r_{n-1} + 2(r_{n-2} + \cdots + r_1),
\]
and
\[
  r_{n+1} = 2n + 1 + r_n + 2(r_{n-1} + r_{n-2} + \cdots + r_1).
\]

Therefore
\[
  r_{n+1} - r_n = 2 + r_n + r_{n-1} \implies r_{n+1} = 2 + 2r_n + r_{n-1}.
\]

Thus
\[
  r_{n+1} + 1 = 2(r_n + 1) + (r_{n-1} + 1),
\]
whence
\[
  r_n + 1 = \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^{n+1},
\]
and
\[
  r_n = \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^{n+1} - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^{n+1} - 1.
\]

Solution 2. Employ the same notation as in Solution 1. We have that \( r_1 = 1 \), \( r_2 = 4 \) and \( r_3 = 11 \). Let \( n \geq 3 \). Consider the situation that there are \( r_{n+1} \) columns. There are basically three types of rook walks.

Type 1. There are four rook walks that enter only the first two columns.

Type 2. There are \( 3r_{n-1} \) rooks walks that do not pass between the second and third columns in the middle row (in either direction), viz. \( r_{n-1} \) of each of the types:
\[
  (1, 1) \to (1, 2) \to (1, 3) \to \cdots \to (3, 3) \to (3, 2) \to (3, 1);
\]
\[
  (1, 1) \to (2, 1) \to (2, 2) \to (1, 2) \to (1, 3) \to \cdots \to (3, 3) \to (3, 2) \to (3, 1);
\]
\[
  (1, 1) \to (1, 2) \to (1, 3) \to \cdots \to (3, 3) \to (3, 2) \to (2, 2) \to (2, 1) \to (3, 1).
\]

Type 3. Consider the rook walks that pass between the second and third column along the middle row.

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They are of Type 3a:

(1, 1) → * → (2, 2) → (2, 3) → ⋅ ⋅ ⋅ → (3, 3) → (3, 2) → (3, 1) ,

or Type 3b:

(1, 1) → (1, 2) → (1, 3) → ⋅ ⋅ ⋅ → (2, 3) → (2, 2) → * → (3, 1) ,

where in each case the asterisk stands for one of two possible options.

We can associate in a two-one way the walks of Type 3a to a rook walk on the last \( n \) columns, namely

(1, 2) → (2, 2) → (2, 3) → ⋅ ⋅ ⋅ → (3, 3) → (3, 2)

and the walks of Type 3b to a rook walk on the last \( n \) columns, namely

(1, 2) → (1, 3) → ⋅ ⋅ ⋅ → (2, 3) → (2, 2) → (3, 2) .

The number of rook walks of the latter two types together is \( r_n - r_{n-1} \). From the number of rook walks on the last \( n \) columns, we subtract one for \( (1, 2) → (2, 2) → (3, 2) \) and \( r_{n-1} \) for those of the type

(1, 2) → (1, 3) → ⋅ ⋅ ⋅ → (3, 3) → (2, 3) .

Therefore, the number of rook walks of Type 3 is \( 2(r_n - 1 - r_{n-1}) \) and we find that

\[
 r_{n+1} = 4 + 3r_{n-1} + 2(r_n - 1 - r_{n-1}) = 2 + 2r_n + r_{n-1} .
\]

We can now complete the solution as in Solution 1.

Solution 3. Let \( S(3,n) \) be the set of self-avoiding rook walks in which the rook occupies column \( n \) but does not occupy column \( n + 1 \). Then \( R(3, n) = |S(3, 1)| + |S(3, 2)| + ⋅ ⋅ ⋅ + |S(3, n)| \). Furthermore, topological considerations allow us to break \( S(3, n) \) into three disjoint subsets \( S_1(3, n) \), the set of paths in which corner \( (1, n) \) is not occupied, but there is a path segment \( (2, n) → (3, n) \); \( S_2(3, n) \), the set of paths in which corners \( (1, n) \) and \( (3, n) \) are both occupied by a path \( (1, n) → (2, n) → (3, n) \); and \( S_3(3, n) \), the set of paths in which corner \( (3, n) \) is not occupied but there is a path segment \( (1, n) → (2, n) \). Let \( s_i(n) = |S_i(3, n)| \) for \( i = 1, 2, 3 \). Note that \( s_1(1) = 0, s_2(1) = 1 \) and \( s_3(1) = 0 \). By symmetry, \( s_1(n) = s_3(n) \) for every positive \( n \). Furthermore, we can construct paths in \( S(3, n + 1) \) by “bulging” paths in \( S(3, n) \), from which we obtain

\[
 s_1(n + 1) = s_1(n) + s_2(n) ;
 s_2(n + 1) = s_1(n) + s_2(n) + s_3(n) ;
 s_3(n + 1) = s_2(n) + s_3(n) ;
\]

or, upon simplification,

\[
 s_1(n + 1) = s_1(n) + s_2(n) ;
 s_2(n + 1) = 2s_1(n) + s_2(n) .
\]

Hence, for \( n ≥ 2 \),

\[
 s_1(n + 1) = s_1(n) + 2s_1(n - 1) + s_2(n - 1)
 = s_1(n) + 2s_1(n - 1) + s_1(n) - s_1(n - 1)
 = 2s_1(n) + s_1(n - 1) .
\]

and

\[
 s_2(n + 1) = 2s_1(n) + s_2(n) = 2s_1(n - 1) + 2s_2(n - 1) + s_2(n)
 = s_2(n) - s_2(n - 1) + 2s_2(n - 1) + s_2(n)
 = 2s_2(n) + s_2(n - 1) .
\]
We find that
\[
s_1(n) = \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^{n-1} - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^{n-1} ;
\]
\[
s_2(n) = \frac{1}{2}(1 + \sqrt{2})^{n-1} + \frac{1}{2}(1 - \sqrt{2})^{n-1}.
\]
Summing a geometric series yields that
\[
R(3, n) = (s_2(1) + \cdots + s_2(n)) + 2(s_1(1) + \cdots + s_1(n))
\]
\[
= \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) \left(\frac{1 + \sqrt{2}}{\sqrt{2}}\right)^n - 1
\]
\[
= \left(\frac{1}{2\sqrt{2}}\right)[(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}] - 1.
\]
The formula agrees with \(R(3, 1) = 1, R(3, 2) = 4\) and \(R(3, 3) = 11.\)

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THE GRADERS’ REPORT

The grading was done by Ed Barbeau, Man-Duen Choi, Felix Recio and Lindsey Shorser. The papers in the top three divisions and most of those in the fourth were marked independently by two markers and all papers in division I were reviewed by four markers, as well as many in division II.

The paper was more difficult this year than in 2007, and it was decided to have but one geometry problem, but that one reasonably easy. There was less opportunity for part marks, and the final results reflect this.

As the papers were being marked, it became evident that there is an increasing gap between the background knowledge and sophistication of the top ten or twenty students and the rest, a matter that should be of some concern to those concerned about an adequate supply of prepared students entering university engineering and science programs. This makes it more difficult to prepare a paper, even for sixty or so invited students, that is a challenge for the top students while being accessible to the rest.

The contest buffs among the students now know as a matter of course results that the reader of this report might not have encountered until their undergraduate or even graduate years, if at all. They know how to solve linear recursion, are familiar with more advanced results of Euclidean geometry such as the theorems of Ceva and Menelaus, or the Ptolemy inequality, are adept at using transformation arguments (including inversion in a circle) in plane geometry, have at their command basic results in graph theory, are familiar with modular arithmetic and number theory results such as Fermat’s Little Theorem and the Chinese Remainder Theorem, know some differential calculus and have a larger supply of inequalities in their quivers, beyond the arithmetic-geometric means and Cauchy-Schwarz inequalities; two students turned to the Muirhead majorization inequalities to solve Question 3. The problems committee did not have most of these advanced techniques in mind in setting the problems; they were designed to be solved through basic reasoning and competent application of standard techniques.

Many students instinctively turn to formulas and computation without taking the time to consider the essence of a problem situation and try an argument that is more attuned to its structure. This was evident in Questions 1, 2 and 4.

Students invited to write the Canadian Mathematical Olympiad cannot expect to do well without preparing for the examination. This should involve reviewing problems of past contests of this level, as well as strengthening their backgrounds in areas in which the school curriculum is weak - geometry, basic combinatorics, inequalities and polynomial algebra. Plenty of material to this end can be obtained through the Canadian Mathematical Society (www.cms.math.ca) and the Mathematical Association of America (www.maa.org).

One flaw in the standard school syllabus became evident in the use either advanced or “clunky” techniques in solving problems. Some students are exposed superficially to more advanced mathematics such as trigonometry and calculus before they have become familiar and facile with more elementary geometry and algebra.

Each problem was marked out of 7.
The marks awarded on the several problems are given in the following table:
Problem 1: The solution mainly relied on the fact that two triangles on the same base and between the same parallels have equal area. Many students, however, cluttered their solutions by inserting altitudes and using the half base-times-height formula; while they usually got it correct, the result was a solution twice as long as it should have been. Several students resorted to analytic geometry, a method particularly cumbersome when the up-and-down formula for the area of a triangle was employed.

Problem 2: Functional equations problems tend to become harder when unwarranted restrictions are imposed on the function, and this situation was no exception. Some students, having had some calculus, reached immediately for the derivative, and got bogged down. They apparently do not realize that not every function admits a derivative. Quite a few students implicitly assumed continuity and drew from the bijective character of $f$ that it was monotone. In another direction, certain candidates took $f$ to be a polynomial, in some cases a linear polynomial, and worked from there. Some solvers were familiar with the result that a function satisfying the equation $g(x + y) = g(x) + g(y)$ on the rationals had the form $g(x) = kx$, and the statement of this fact without a proof was accepted by the markers.

Problem 3: Performance on this problem was better than expected, and several effective methods were adduced to handling it. While an astute application of the arithmetic-geometric means or harmonic-arithmetic means inequality sufficed to bring it home, a few candidates brought much heavier machinery into play without making the solution any more efficient.

Problem 4: This was a tricky problem, but not done as well as expected. The possibility of congruence modulo 1 did not cause any difficulty except for one or two cases. Many candidates were familiar with the little Fermat theorem (and this should be regarded as an essential part of the preparation for the CMO); nevertheless, fewer identified the solutions $f(n) = n$ or $f(p) = 1$ for all primes $p$ than we had hoped. Most of the candidates who gained points on this problem did derive early on that $f(p)$ divided $p$ and so had to be either 1 or $p$, for each prime $p$. A few thought that one of these alternatives always had to occur. Quite a few set $f(n) = n + kp$, at which point the solutions generally became hopelessly complicated.

Problem 5: This was intended to be a hard problem, and it did require the students to solve a linear recursion. However, four marks were available to any student who could set up the recursion; the main burden of the problem was the analysis of the different types of rook walks and the formulation of an induction process. We were disappointed at how few students were able to reason carefully enough to get even a good start at this. A couple of candidates misread the problem and had the walk ending in the upper-right corner.