Report of the Thirty Fifth
Canadian Mathematical Olympiad
2003

Canadian Mathematical Society
Société mathématique du Canada

Sun Life Financial

The UNIVERSITY of WINNIPEG

The Canadian Mathematical Olympiad (CMO) is an annual national mathematics competition sponsored by the Candian Mathematical Society (CMS) and is administered by the Canadian Mathematical Olympiad Committee (CMO Committee), a sub-committee of the Mathematical Competitions Committee. The CMO was established in 1969 to provide an opportunity for students who performed well in various provincial mathematics competitions to compete at a national level. It also serves as preparation for those Canadian students competing at the International Mathematical Olympiad (IMO).

Students qualify to write the CMO by earning a sufficiently high score on the Canadian Open Mathematical Challenge (COMC). Students may also be nominated to write the CMO by a provincial coordinator.

The Society is grateful for support from the Sun Life Assurance Company of Canada as the Major Sponsor of the 2003 Canadian Mathematical Olympiad and the other sponsors which include: the Ministry of Education of Ontario, the Ministry of Education of Quebec, Alberta Learning, the Department of Education of New Brunswick, the Department of Education of Newfoundland and Labrador, the Department of Education of the Northwest Territories and the Department of Education of Saskatchewan; the Department of Mathematics and Statistics, University of Winnipeg; the Department of Mathematics and Statistics, University of New Brunswick at Fredericton; the Centre for Education in Mathematics and Computing, University of Waterloo; the Department of Mathematics and Statistics, University of Ottawa; the Department of Mathematics, University of Toronto; Nelson Thompson Learning; A.K. Peter, Ltd.; and John Wiley and Sons Canada Ltd.

The provincial coordinators of the CMO are Peter Crippin, University of Waterloo ON; John Denton, Dawson College QC; Diane Dowling, University of Manitoba; Harvey Gerber, Simon Fraser University BC; Gareth J. Griffith, University of Saskatchewan; Jacques Labelle, Université du Québec à Montréal; Ted Lewis, University of Alberta; Gordon MacDonald, University of Prince Edward Island; Roman Mureika, University of New Brunswick; Michael Nutt, Acadia University NS; Thérèse Ouellet, Université de Montréal QC; Donald Rideout, Memorial University of Newfoundland.

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Terry Visentin, Chair
Canadian Mathematical Olympiad Committee

The 35th (2003) Canadian Mathematical Olympiad was held on Wednesday, March 26th, 2003 with 85 competitors from 55 schools in nine Canadian provinces participating. One Canadian student wrote the exam in the United States. The number of contestants from each province was as follows:

BC(11) AB(6) SK(1) MB(3) ON(55) QC(5) NB(1) NS(1) NF(1)

The 2003 CMO consisted of five questions. Each question was worth 7 marks for a total maximum score of m=35. The contestants performances were grouped into four divisions as follows:

Division	**Range of Scores**	**No. of Students**
I	$25 \leq m \leq 35$	9
II	$20 \leq m < 25$	12
III	$14 \leq m < 20$	26
IV	$0 \leq m < 14$	38

FIRST PRIZE— Sun Life Financial Cup — $2000
János Kramár
University of Toronto Schools, Toronto, Ontario

SECOND PRIZE — $1500
Tianyi (David) Han
Woburn Collegiate Institute, Toronto, Ontario

THIRD PRIZE — $1000
Robert Barrington Leigh
Old Scona Academic High School, Edmonton, Alberta.

HONOURABLE MENTIONS — $500

Olena Bormashenko
Don Mills Collegiate Institute, Toronto, Ontario

Ali Feizmohammadi
Northview Heights Secondary School, North York, Ontario

Ralph Furmaniak
A.B. Lucas Secondary School, London, Ontario

Oleg Ivrii
Don Mills Collegiate Institute, Don Mills, Ontario

Andrew Mao
A.B. Lucas Secondary School, London, Ontario

Jacob Tsimerman
University of Toronto Schools, Toronto, Ontario

Division 2

$20 \le m < 25$

Aaron Chan	J.N. Burnett S.S.	BC
Justin Chan	Mount Douglas S.S.	BC
Leonid Chindelevitch	Marianopolis College	QC
Joe Hung	David Thompson S.S.	BC
Ruohan Li	Forest Hill C.I.	ON
Razvan Romanescu	East York C.I.	ON
Samuel Wong	University Hill S.S.	BC
Nan Yang	Birchmount Park C.I.	ON
Dongbo Yu	Don Mills C.I.	ON
Matei Zaharia	Jarvis C.I.	ON
Lingkai Zeng	Edison H.S.	NJ
Yin Zhao	Vincent Massey S.S.	ON

Division 3

$14 \le m < 20$

Billy Ballik	Bowmanville H.S.	ON
David Belanger	Nicholson Catholic College	ON
Robert Biswa S	Vincent Massey S.S.	ON
Maximilian Butler	Tom Griffiths Home School	ON
Francis Chung	A.B. Lucas S.S.	ON
Andrew Critch	Clarenville Integrated H.S.	NF
Rong Tao Dan	Point Grey S.S.	BC
Gabriel Gauthier	Marianopolis College	QC
Chen Huang	Sir Winston Churchill S.S.	BC
Liji Huang	Portage Collegiate Institute	MB
Jenny Yue Jin	Earl Haig S.S.	ON
Hyon Lee	Vincent Massey S.S.	ON
Charles Zhi Li	Western Canada H.S.	AB
Robin Li	Ontario Science Centre	ON
David Rhee	Vernon Barford Junior High	AB
Chen Shen	A.Y. Jackson S.S.	ON
Jimmy Shen	Vincent Massey S.S.	ON
Evan Stratford	University Of Toronto Schools	ON
John Sun	Vincent Massey S.S.	ON
Yang Yang	Don Mills C.I.	ON
Ti Yin	William Lyon Mackenzie C.I.	ON
Tom Yue	A.Y. Jackson S.S.	ON
Hang Zhang	Albert Campbell C.I.	ON
Nancy Zhang	Sir Winston Churchill S.S.	BC
Peter Zhang	Sir Winston Churchill H.S.	AB
Zhongying Zhou	Vincent Massey S.S.	ON

Division 4

$0 \le m < 14$

Allan Cai	Bayview Secondary School	ON
Alex Chou	Semiahmoo S.S.	BC
Johnston Chu	Earl Haig S.S.	ON
Eric Dallal	Marianopolis College	QC
Fan Feng	Vincent Massey S.S.	ON
Nir Friedman	Thornhill S.S.	ON
Jeremy Green	Richview C.I.	ON
Mathieu Guay Paquet	College De Maisonneuve	QC
Ian Hung	University Of Toronto Schools	ON
Alexander Jiang	The Halifax Grammar School	NS
Heejune Jun	St. Robert C.H.S.	ON
Keigo Kawaji	Earl Haig S.S.	ON
Jaeseung Kim	Bayview Secondary School	ON
Kanguk Lee	Newtonbrook S.S.	ON
Tae Hun Lee	Carson Graham S.S.	BC
Angela Lin	Sir Winston Churchill S.S.	BC
Nanqian Lin	Albert Campbell C.I.	ON
David Liu	St. John's-Ravenscourt School	MB
Taotao Liu	Vincent Massey S.S.	ON
Rui Ma	Vincent Massey S.S.	ON
Radoslav Marinov	Harry Ainlay H.S.	AB
Andre Mutchnik	Marianopolis College	QC
Quoc Nguyen	Vincent Massey Collegiate	MB
Jennifer Park	Bluevale C.I.	ON
Mark Salzman	Upper Canada College	ON
Antonio Sanchez	St. Matthew H.S.	ON
Ner Mu Nar Saw	Jarvis C.I.	ON
Yi Hao Shen	Saint John H.S.	NB
Sarah Sun	Holy Trinity Academy	AB
Yuanbin Tang	Lisgar C.I.	ON
William Truong	Campbell C.I.	SK
Xingfang Wang	Kitchener-Waterloo C.I. & V.I.	ON
Yifei Wang	Vincent Massey S.S.	ON
Yehua Wei	York Mills C.I.	ON
Shaun White	Vincent Massey S.S.	ON
Yang Xia	Vincent Massey S.S.	ON
Zhengzheng Yang	The Woodlands S.	ON
Jeff Zhao	Eric Hamber S.S.	BC

The Canadian Mathematical Olympiad - 2003

Wednesday, March 26

1. Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \leq m \leq 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

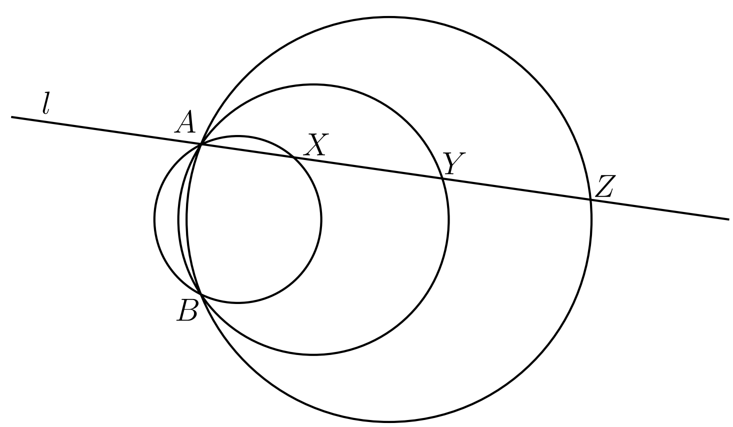
2. Find the last three digits of the number $2003^{2002^{2001}}$.

3. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$

$$x^2 + y^2 + z^2 = xyz.$$

4. Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



5. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Solutions to the 2003 CMO

written March 26, 2003

1. Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \le m \le 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

Solution

The minute hand makes a full revolution of 360° every 60 minutes, so after m minutes it has swept through $\frac{360}{60}m = 6m$ degrees. The hour hand makes a full revolution every 12 hours (720 minutes), so after m minutes it has swept through $\frac{360}{720}m = m/2$ degrees. Since both hands started in the same position at 12:00, the angle between the two hands will be 1° if $6m - m/2 = \pm 1 + 360k$ for some integer k. Solving this equation we get

$$m = \frac{720k \pm 2}{11} = 65k + \frac{5k \pm 2}{11}.$$

Since $1 \le m \le 720$, we have $1 \le k \le 11$. Since m is an integer, $5k \pm 2$ must be divisible by 11, say $5k \pm 2 = 11q$. Then

$$5k = 11q \pm 2 \quad \Rightarrow \quad k = 2q + \frac{q \pm 2}{5}.$$

If is now clear that only $q = 2$ and $q = 3$ satisfy all the conditions. Thus $k = 4$ or $k = 7$ and substituting these values into the expression for m we find that the only possible values of m are 262 and 458.

2. Find the last three digits of the number $2003^{2002^{2001}}$.

Solution

We must find the remainder when $2003^{2002^{2001}}$ is divided by 1000, which will be the same as the remainder when $3^{2002^{2001}}$ is divided by 1000, since $2003 \equiv 3 \pmod{1000}$. To do this we will first find a positive integer n such that $3^n \equiv 1 \pmod{1000}$ and then try to express 2002^{2001} in the form $nk + r$, so that

$$2003^{2002^{2001}} \equiv 3^{nk+r} \equiv (3^n)^k \cdot 3^r \equiv 1^k \cdot 3^r \equiv 3^r \pmod{1000}.$$

Since $3^2 = 10 - 1$, we can evaluate 3^{2m} using the binomial theorem:

$$3^{2m} = (10-1)^m = (-1)^m + 10m(-1)^{m-1} + 100\frac{m(m-1)}{2}(-1)^{m-2} + \cdots + 10^m.$$

After the first 3 terms of this expansion, all remaining terms are divisible by 1000, so letting $m = 2q$, we have that

$$3^{4q} \equiv 1 - 20q + 100q(2q-1) \pmod{1000}. \tag{1}$$

Using this, we can check that $3^{100} \equiv 1 \pmod{1000}$ and now we wish to find the remainder when 2002^{2001} is divided by 100.

Now $2002^{2001} \equiv 2^{2001} \pmod{100} \equiv 4 \cdot 2^{1999} \pmod{4 \cdot 25}$, so we'll investigate powers of 2 modulo 25. Noting that $2^{10} = 1024 \equiv -1 \pmod{25}$, we have

$$2^{1999} = (2^{10})^{199} \cdot 2^9 \equiv (-1)^{199} \cdot 512 \equiv -12 \equiv 13 \pmod{25}.$$

Thus $2^{2001} \equiv 4 \cdot 13 = 52 \pmod{100}$. Therefore 2002^{2001} can be written in the form $100k + 52$ for some integer k, so

$$2003^{2002^{2001}} \equiv 3^{52} \pmod{1000} \equiv 1 - 20 \cdot 13 + 1300 \cdot 25 \equiv 241 \pmod{1000}$$

using equation (1). So the last 3 digits of $2003^{2002^{2001}}$ are 241.

3. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$

$$x^2 + y^2 + z^2 = xyz.$$

Solution 1

Let $f(x, y, z) = (x^3 - x) + (y^3 - y) + (z^3 - z)$. The first equation above is equivalent to $f(x, y, z) = 0$. If $x, y, z \geq 1$, then $f(x, y, z) \geq 0$ with equality only if $x = y = z = 1$. But if $x = y = z = 1$, then the second equation is not satisfied. So in any solution to the system of equations, at least one of the variables is less than 1. Without loss of generality, suppose that $x < 1$. Then

$$x^2 + y^2 + z^2 > y^2 + z^2 \geq 2yz > yz > xyz.$$

Therefore the system has no real positive solutions.

Solution 2

We will show that the system has no real positive solution. Assume otherwise.

The second equation can be written $x^2 - (yz)x + (y^2 + z^2)$. Since this quadratic in x has a real solution by hypothesis, its discrimant is nonnegative. Hence

$$y^2z^2 - 4y^2 - 4z^2 \geq 0.$$

Dividing through by $4y^2z^2$ yields

$$\frac{1}{4} \geq \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{y^2}.$$

Hence $y^2 \geq 4$ and so $y \geq 2$, y being positive. A similar argument yields $x, y, z \geq 2$. But the first equation can be written as

$$x(x^2 - 1) + y(y^2 - 1) + z(z^2 - 1) = 0,$$

contradicting $x, y, z \geq 2$. Hence, a real positive solution cannot exist.

Solution 3

Applying the arithmetic-geometric mean inequality and the Power Mean Inequalities to x, y, z we have

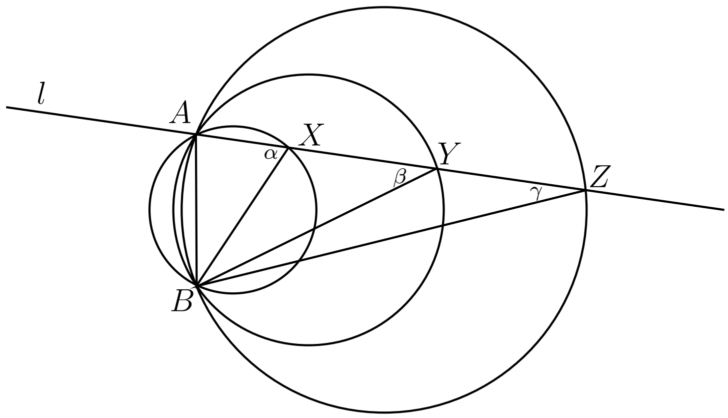
$$\sqrt[3]{xyz} \le \frac{x+y+z}{3} \le \sqrt{\frac{x^2+y^2+z^2}{3}} \le \sqrt[3]{\frac{x^3+y^3+z^3}{3}}.$$

Letting $S = x+y+z = x^3+y^3+z^3$ and $P = xyz = x^2+y^2+z^2$, this inequality can be written

$$\sqrt[3]{P} \le \frac{S}{3} \le \sqrt{\frac{P}{3}} \le \sqrt[3]{\frac{S}{3}}.$$

Now $\sqrt[3]{P} \le \sqrt{\frac{P}{3}}$ implies $P^2 \le P^3/27$, so $P \ge 27$. Also $\frac{S}{3} \le \sqrt[3]{\frac{S}{3}}$ implies $S^3/27 \le S/3$, so $S \le 3$. But then $\sqrt[3]{P} \ge 3$ and $\sqrt[3]{\frac{S}{3}} \le 1$ which is inconsistent with $\sqrt[3]{P} \le \sqrt[3]{\frac{S}{3}}$. Therefore the system cannot have a real positive solution.

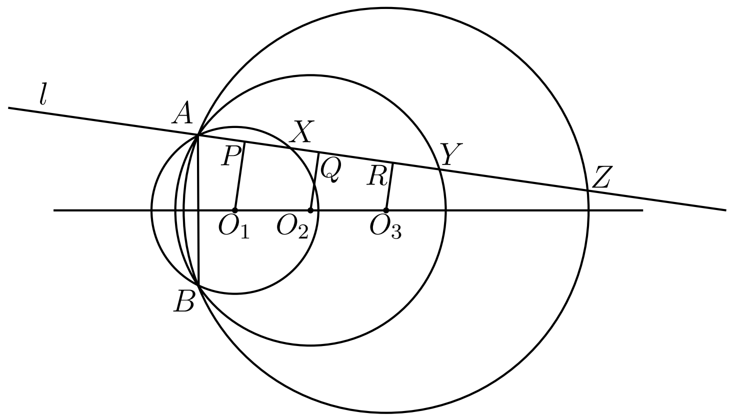
4. Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



Solution 1

Let l be a line through A different from AB and join B to A, X, Y and Z as in the above diagram. No matter how l is chosen, the angles AXB, AYB and AZB always subtend the chord AB. For this reason the angles in the triangles BXY and BXZ are the same for all such l. Thus the ratio $XY : YZ$ remains constant by similar triangles.

Note that this is true no matter how X, Y and Z lie in relation to A. Suppose X, Y and Z all lie on the same side of A (as in the diagram) and that $\measuredangle AXB = \alpha$, $\measuredangle AYB = \beta$ and $\measuredangle AZB = \gamma$. Then $\measuredangle BXY = 180° - \alpha$, $\measuredangle BYX = \beta$, $\measuredangle BYZ = 180° - \beta$ and $\measuredangle BZY = \gamma$. Now suppose l is chosen so that X is now on the opposite side of A from Y and Z. Now since X is on the other side of the chord AB, $\measuredangle AXB = 180° - \alpha$, but it is still the case that $\measuredangle BXY = 180° - \alpha$ and all other angles in the two pertinent triangles remain unchanged. If l is chosen so that X is identical with A, then l is tangent to the first circle and it is still the case that $\measuredangle BXY = 180° - \alpha$. All other cases can be checked in a similar manner.



Solution 2

Let m be the perpendicular bisector of AB and let O_1, O_2, O_3 be the centres of the three circles. Since AB is a chord common to all three circles, O_1, O_2, O_3 all lie on m. Let l be a line through A different from AB and suppose that X, Y, Z all lie on the same side of AB, as in the above diagram. Let perpendiculars from O_1, O_2, O_3 meet l at P, Q, R, respectively. Since a line through the centre of a circle bisects any chord,

$$AX = 2AP, \quad AY = 2AQ \quad \text{and} \quad AZ = 2AR.$$

Now

$$XY = AY - AX = 2(AQ - AP) = 2PQ \quad \text{and, similarly,} \quad YZ = 2QR.$$

Therefore $XY : YZ = PQ : QR$. But $O_1P \,||\, O_2Q \,||\, O_3R$, so $PQ : QR = O_1O_2 : O_2O_3$. Since the centres of the circles are fixed, the ratio $XY : YZ = O_1O_2 : O_2O_3$ does not depend on the choice of l.

If X, Y, Z do not all lie on the same side of AB, we can obtain the same result with a similar proof. For instance, if X and Y are opposite sides of AB, then we will have $XY = AY + AX$, but since in this case $PQ = AQ + AP$, it is still the case that $XY = 2PQ$ and result still follows, etc.

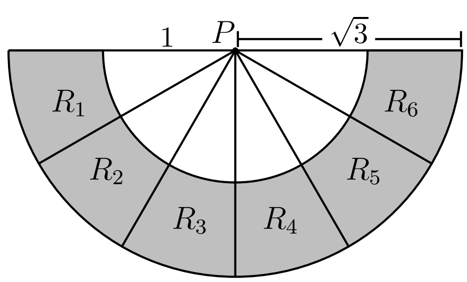
5. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

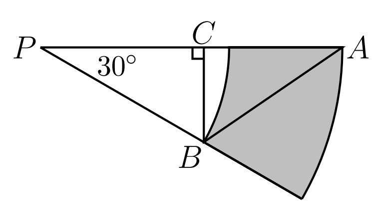
Solution

We will construct the set T in the following way: Assume the points of S are in the xy-plane and let P be a point in S with maximum y-coordinate. This point P will be a member of the set T and now, from S, we will remove P and all points in S which are less than $\sqrt{3}$ units from P. From the remaining points we choose one with maximum y-coordinate to be a member of T and remove from S all points at distance less than $\sqrt{3}$ units from this new point. We continue in this way, until all the points of S are exhausted. Clearly any two points in T are at least $\sqrt{3}$ units apart. To show that T has at least $n/7$ points, we must prove that at each stage no more than 6 other points are removed along with P.

At a typical stage in this process, we've selected a point P with maximum y-coordinate, so any points at distance less than $\sqrt{3}$ from P must lie inside the semicircular region of radius $\sqrt{3}$ centred at P shown in the first diagram below. Since points of S are at least 1 unit apart, these points must lie outside (or on) the semicircle of radius 1. (So they lie in the shaded region of the first diagram.) Now divide this shaded region into 6 congruent regions $R_1, R_2, \ldots, R_6$ as shown in this diagram.

We will show that each of these regions contains at most one point of S. Since all 6 regions are congruent, consider one of them as depicted in the second diagram below. The distance between any two points in this shaded region must be less than the length of the line segment AB. The lengths of PA and PB are $\sqrt{3}$ and 1, respectively, and angle $APB = 30°$. If we construct a perpendicular from B to PA at C, then the length of PC is $\cos 30° = \sqrt{3}/2$. Thus BC is a perpendicular bisector of PA and therefore $AB = PB = 1$. So the distance between any two points in this region is less than 1. Therefore each of $R_1, \ldots, R_6$ can contain at most one point of S, which completes the proof.





GRADER'S REPORT

Each question was worth a maximum of 7 marks. Every solution on every paper was graded by two different markers. If the two marks differed by more than one point, the solution was reconsidered until the difference was resolved. If the two marks differed by one point, the average was used in computing the total score. The top papers were then reconsidered until the committee was confident that the prize-winning contestants were ranked correctly. The various marks assigned to each solution are displayed below, as a percentage. As described above, fractional scores are possible, but for the purpose of this table, marks are rounded up. So, for example, 56.5% of the students obtained a score of 6.5 or 7 on the first problem. This indicates that on 56.5% of the papers, at least one marker must have awarded a 7 on question #1.

Marks	**#1**	**#2**	**#3**	**#4**	**#5**
0	1.2	9.4	34.1	40.0	52.9
1	7.1	29.4	14.1	12.9	30.6
2	3.5	20.0	7.1	8.2	9.4
3	10.6	4.7	0.0	0.0	1.2
4	4.7	4.7	0.0	0.0	0.0
5	7.1	7.1	2.4	3.5	0.0
6	9.4	5.9	10.6	14.1	3.5
7	56.5	18.8	31.8	21.2	2.4

At the outset our marking philosophy was as follows: A score of 7 was given for a completely correct solution. A score of 6 indicated a solution which was essentially correct, but with a very minor error or omission. Very significant progress had to be made to obtain a score of 3. Even scores of 1 or 2 were not awarded unless some significant work was done. Scores of 4 and 5 were reserved for special situations. This worked very well for the last 3 problems where students who took the right approach tended to completely solve the problem. But in Problems 1 and 2, where many more degrees of partial progress occurred, the marking schemes were adjusted a bit and are described separately.

PROBLEM 1

This problem was very well done. Almost all students correctly set up equations which described the situation and for this they received 3 points. The remaining 4 points were awarded for correctly finding the integer solutions. There was no approach taken that was markedly different from the official solution, although some students were very efficient about solving the equation using modular arithmetic while others exhaustively searched many possibilities. Some students split m up into hours and minutes, but this doesn't really alter the solution very much.

PROBLEM 2

This problem was the most difficult to grade. Students received a mark for finding a power n such that $2003^n \equiv 1$ mod 1000 and a further mark for realizing that they now must reduce 2002^{2001} mod n. None of the correct solutions were very different from the official solutions although many students did a lot of computation of various powers rather than invoking the binomial theorem and a fair number of numerical errors occurred. Students who knew Euler's Theorem could immediately write down $2003^{400} \equiv 1$ mod 1000, since $\phi(1000)=400$, but then had to reduce 2002^{2001} modulo 400 instead of 100.

PROBLEM 3

A wide variety of approaches were taken to solve this problem. A few students solved it using very elementary methods like those of the first two official solutions. Most invoked AM-GM or Cauchy-Schwarz inequalities and several students used some ideas from calculus. One student used the Chebyshev inequality. Students who surmised that the system had no solution and started out with a reasonable approach were usually able to successfully prove their assertion.

PROBLEM 4

This problem was also solved using quite a few different methods and most students either knew exactly how to solve it or didn't make very much progress at all. Students who only considered the case where X,Y,Z all lie on the same side of A (as in the diagram) received a maximum score of 6 on this problem. To obtain a score of 7, a student had to at least mention that other configurations are possible and describe how their method would have to be adjusted to handle other cases. Students should be aware that diagrams are included to enhance the reading of the question and to encourage a consistent use of notation, but can't depict all of the possibilities. It's the written statement of the problem that must be followed.

Besides the two official solutions, a common approach was to consider a fixed line through B intersecting the circles at P,Q,R, say, and then show that XY:YZ=PQ:QR. This approach is successful and most students using it received a score of 6, but it was not included in the official solutions because it's more cumbersome to deal with all of the possible configurations which might arise.

Several students were successful using analytic geometry. A typical approach fixed A and B on one axis symmetrically placed about the origin, and had the centres of the circles placed on the other axis with intercepts a,b,c. The calculations get rather messy, but the few who used this method managed to show that XY:YZ=b–a:c–b and didn't have to deal with additional cases.

PROBLEM 5

Few students made significant progress on this challenging problem. The five students who attained a mark of 6 or 7 all used the same approach as the one used in the official solutions. The key idea seems to be to construct T algorithmically by beginning with some point on the convex hull of S. Students who tried to divide the plane up into regions in some way and invoke the pigeonhole principle were unsuccessful, but obtained 1 or 2 points for this approach.