Artin-Nagata Properties, Minimal Multiplicities, and Depth of Fiber Cones

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Reference

Artin-Nagata properties, minimal multiplicities, and depth of fiber cones, Jonathan Montaño, to appear in Journal of Algebra, arXiv:1408.5541.

Setting

- $(R, \mathfrak{m}, k)$ Cohen-Macaulay (CM), $|k| = \infty$, $\dim(R) = d > 0$.
- I an R-ideal of height $h > 0$.
- $J \subseteq I$ is a minimal reduction of I, i.e., $I^{n+1} = JI^n$ for some $n \in \mathbb{N}$ and J is minimal with respect to inclusion.
- $r(I) = \min\{n \mid I^{n+1} = JI^n,\ \text{for some minimal reduction } J\}$, the reduction number of I.

Blowup algebras

- $\mathcal{R}(I) = \bigoplus_{n\geqslant 0} I^n$, the Rees algebra of I.
- $\mathcal{G}(I) = \bigoplus_{n\geqslant 0} I^n/I^{n+1}$, the associated graded algebra of I.
- $\mathcal{F}(I) = \bigoplus_{n\geqslant 0} I^n/\mathfrak{m}I^n$, the fiber cone of I.

Dimension:

- $\dim \mathcal{R}(I) = d + 1$
- $\dim \mathcal{G}(I) = d$.
- $\ell := \dim \mathcal{F}(I)$, the analytic spread of I.

How do the depths of the blowup algebras relate?

- $\mathcal{R}(I)$ is CM $\Rightarrow \mathcal{G}(I)$ is CM. (Huneke)
- $\mathcal{R}(I)$ is CM $\Leftrightarrow \mathcal{G}(I)$ is CM and $a(\mathcal{G}(I)) < 0$. (Ikeda-Trung)
- If R is regular, $\mathcal{R}(I)$ is CM $\Leftrightarrow \mathcal{G}(I)$ is CM. (Lipman)
- If I is $\mathfrak{m}$-primary, $\mathcal{R}(I)$ is CM $\Leftrightarrow \mathcal{G}(I)$ is CM and $r(I) < d$. (Goto-Shimoda)
- $\mathcal{G}(I)$ is not CM $\Rightarrow$ depth $\mathcal{R}(I) =$ depth $\mathcal{G}(I) + 1$ (Huckaba-Marly)

However, in general:

- $\mathcal{F}(I)$ is CM $\not\Rightarrow \mathcal{G}(I)$ is CM.
- $\mathcal{F}(I)$ is CM $\not\Leftarrow \mathcal{R}(I)$ is CM.

Hilbert-Samuel multiplicity

Let I be $\mathfrak{m}$-primary,

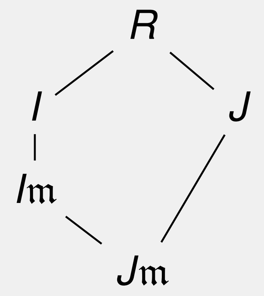
$$e(I) = (d-1)! \lim_{n\to\infty} \frac{\lambda(I^n/I^{n+1})}{n^{d-1}}$$

is the Hilbert-Samuel multiplicity of I.

Minimal multiplicity ($\mathfrak{m}$-primary case)

Notions of minimal multiplicity provide stronger relations between the depths blowup algebras:

Let I be $\mathfrak{m}$-primary. From the following diagram



we obtain

$$e(I) \geqslant \mu(I) - d + \lambda(R/I) \tag{1}$$

with equality iff $I\mathfrak{m} = J\mathfrak{m}$.

- I is of Goto-minimal multiplicity ("=" in (1)): $\mathcal{R}(I)$ is CM $\Leftrightarrow \mathcal{G}(I)$ is CM $\Leftrightarrow r(I) \leqslant 1$. (Goto)

Question: How can we define a notion of minimal multiplpicty for non $\mathfrak{m}$-primary ideals?

j-multiplicity

Let I be any ideal,

$$j(I) = (d-1)! \lim_{n\to\infty} \frac{\lambda(\mathrm{H}^0_{\mathfrak{m}}(I^n/I^{n+1}))}{n^{d-1}}$$

is the j-multiplicity (Achilles-Manaresi).

With the j-multiplicity several results for $\mathfrak{m}$-primary ideals have been extended to arbitrary ideals using the j-multiplicity instead of the Hilbert-Samuel multiplicity. For example:

- Teissier's volume interpretation of multiplicities of monomial ideals. (Jeffries-M)
- Rees criterion of integral dependence. (Flenner-Manaresi)
- Relation with depths of blowup algebras. (Polini-Xie, Mantero-Xie, M)

Artin-Nagata properties

Recall $\ell = \ell(I)$ and $h = \mathrm{ht}(I)$.

Assumption $(\star)$

The following ideals satisfy the Artin-Nagata properties $AN_{\ell-2}$ and G_ℓ:

- Ideals with $\ell = h$.
- Ideals with $\ell = h + 1$ that are generically a complete intersection.
- Ideals with $\mu(I) \leqslant h + 2$.
- Perfect height 2 and perfect Gorenstein height 3 ideals that are a complete intersection locally in Spec $R \setminus \{\mathfrak{m}\}$.

Minimal multiplicity (general case)

Theorem (Achilles-Manaresi, Xie)
Let $x_1, \ldots, x_{d-1}$ be $d-1$ general elements in I and $\widetilde{R} := R/(x_1, \ldots, x_{d-1}) : I^\infty$. Then $\dim \widetilde{R} \leqslant 1$, the ideal $\widetilde{I} := I\widetilde{R}$ is $\widetilde{\mathfrak{m}}$-primary, and

$$j(I) = e(\widetilde{I})$$

.

From this theorem and (1) we obtain:

$$j(I) \geqslant \lambda(\widetilde{R}/\widetilde{I}) + \mu(\widetilde{I}) - 1. \tag{2}$$

Proposition (M)
Under $(\star)$, I is of Goto-minimal j-multiplicity ("=" in (2)) $\Leftrightarrow I\mathfrak{m} = J\mathfrak{m}$, for one (hence every) J.

Results

Theorem 1: (M)
Under $(\star)$. Assume $I\mathfrak{m} = J\mathfrak{m}$, consider the following statements:
(i) $\mathcal{R}(I)$ is CM,
(ii) $\mathcal{G}(I)$ is CM,
(iii) $\mathcal{F}(I)$ is CM and $a(\mathcal{F}(I)) \leqslant -h + 1$,
(iv) $r(I) \leqslant \ell - h + 1$.

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).
If in addition depth $R/I^j \geqslant d - h - j + 1$ for every $1 \leqslant j \leqslant \ell - h + 1$, then all the statements are equivalent.

Theorem 2: (M)
Under $(\star)$. Assume $J \cap I^n\mathfrak{m} = JI^{n-1}\mathfrak{m}$ for every $2 \leqslant n \leqslant r(I)$, then TFAE:
(i) $\mathcal{F}(I)$ is CM.
(ii) depth $\mathcal{G}(I) \geqslant \ell - 1$.
(iii) depth $\mathcal{R}(I) \geqslant \ell$.

Examples

1) The monomial ideals

$$I = (x_1^2, x_1x_2, \ldots, x_1x_d, x_2^2, x_2x_3, \ldots, x_2x_n)$$

are strongly stable ideals of height 2 satisfying $(\star)$. I is of Goto-minimal j-multiplicity then the algebras $\mathcal{R}(I)$, $\mathcal{G}(I)$, and $\mathcal{F}(I)$ are CM.

2) Let $R = k[[x, y, z, w]]$ and

$$M = \begin{pmatrix} x & y & z & w \\ w & x & y & z \end{pmatrix}.$$

The ideal $I = I_2(M)$ has height 3, satisfies $(\star)$, $r(I) \leqslant 2$, and R/I is CM. I is of Goto-minimal j-multiplicity then the algebras $\mathcal{R}(I)$, $\mathcal{G}(I)$, and $\mathcal{F}(I)$ are CM.