On the minimal degree of faithful representations of elementary Chevalley groups over $\mathbb{Z}/p^n\mathbb{Z}$

Camelia Karimianpour

Department of Mathematics and Statistics, University of Ottawa

Abstract

We determine a lower bound for the degree of faithful representations of Chevalley groups over $\mathbb{Z}/p^n\mathbb{Z}$. Our key step is a use of the Stone–von Neumann theorem.

Notations

- $R := \mathbb{Z}/p^n\mathbb{Z}$.
- $\mathbf{G}_{ad}$: an elementary (adjoint) Chevalley group corresponding to the irreducible root system Φ.
- $\Sigma^+ := \{\alpha \in \Phi^+ : \langle \alpha, \tilde{\beta} \rangle = 1\}$, where $\tilde{\beta}$ is the highest root.
- H: a Heisenberg subgroup of $\mathbf{G}_{ad}$.
- A: a maximal abelian subgroup of H.
- (ρ, V): a faithful (finite dimensional) representation of $\mathbf{G}_{ad}$:

  $\rho : \mathbf{G}_{ad} \to \mathrm{GL}(V)$ is an injective group homomorphism.
- $\rho^{\lambda,\alpha}$: conjugation of $\rho|_H$ by $h_{\lambda,\alpha}$ in the normalizer of H.
- χ_1: the central character of an irreducible representation (ρ_1, V_1) of H.
- $\mathrm{Ind}_A^H \chi$: induced representation of an extension of χ_1 to H.
- φ: Euler function.

Problem

We determine a lower bound for the smallest possible dimension of a faithful representation (ρ, V) of a (Chevalley) group $\mathbf{G}_{ad}(R)$.

Our Method

Given a faithful representation (ρ, V) of a Chevalley group $\mathbf{G}_{ad}$, we show that $\rho|_H$ has at least one irreducible constituent (σ_1, V_1) with a faithful central character. Applying a version of the Stone-von Neumann theorem, we show that the dimension of V_1 is $[H : A] = p^{dn}$, where $d = |\Sigma^+|/2$. The lower bound for $\dim(V)$ is obtained from the calculation of the size of the orbit of the action of the normalizer of H on (σ_1, V_1).

Stone-von Neumann Theorem

Let H be a finite Heisenberg group with center $Z(H)$,

- let $\chi_\circ$ be a one-dimensional representation of $Z(H)$ such that $\ker(\chi_\circ) \cap [H,H] = \{1\}$.
- let χ be any extension of $\chi_\circ$ to a maximal abelian subgroup A,

Then

- up to isomorphism, $\mathrm{Ind}_A^H \chi$ is the unique irreducible representation of H with central character $\chi_\circ$.

Step 1

- Restriction of ρ to H decomposes into irreducible representations:
  $$\rho|_H = V_1 \oplus \cdots \oplus V_m$$
- One of the irreducible constituents (say (ρ_1, V_1)) has a faithful central character χ_1 and hence satisfies the Stone-von Neumann theorem
- It is isomorphic to $\mathrm{Ind}_A^H \chi_1$.
- $\dim(V_1) = \dim \mathrm{Ind}_A^H \chi_1 = [H : A]$.

Step 2

- For each $\alpha \in \Phi$ and $\lambda \in R^*$, we construct an element $h_{\lambda,\alpha}$ in the normalizer of H.
- $\rho \cong \rho^{\lambda,\alpha}$ as H-modules implies that $\rho_1^{\lambda,\alpha}$ occurs among the irreducible components of $\rho|_H$.
- We count the number N of distinct $\rho_1^{\lambda,\alpha}$ by counting the number of distinct central characters $\chi_1^{\lambda,\alpha}$.
- We deduce that $\dim(\rho) \geq N[H : A]$.

Remark

1. The root system Φ is not A_1 or C_ℓ for $\ell \geq 2$: there exists a root in Φ such that the $\chi_1^{\lambda,\alpha}$ are distinct for all $\lambda \in R^*$ $\rightsquigarrow$ there are $|R^*| = \varphi(p^n)$ distinct characters.
2. The root system Φ is either C_ℓ or A_1: we can only construct $\frac{1}{2}|R^*| = \frac{1}{2}\varphi(p^n)$ distinct characters.

Example: $\mathbf{G}_{ad} = \mathrm{SL}_k(\mathbb{Z}/p^n\mathbb{Z})$

$$H = \begin{pmatrix} 1 & * & * \\ 0 & I_{k-2} & * \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & * & * \\ 0 & I_{k-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z(H) = \begin{pmatrix} 1 & 0 & * \\ 0 & I_{k-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that $Z(H) \cong R$, and $[H, A] = p^{k-2}$. Fix $\alpha \in \Phi$.

$$h_\lambda = \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & I_{k-2} \end{pmatrix}, \quad \lambda \in R^*, \quad, \chi_1{}^\lambda(x) = \chi_1(\lambda x) \;\; x \in R$$

There are $|R^*|$ distinct characters $\chi_1^{h_\lambda}$, thus
$$\dim(\rho) \geq \varphi(p^n) p^{k-2}$$

Theorem

Let $\mathrm{Rep}_f(\mathbf{G}_{ad})$ denote the set of all finite dimensional faithful representations of $\mathbf{G}_{ad}$ over complex vector spaces, and set $m_f(\mathbf{G}_{ad}) := \min\{\deg(\rho) : \rho \in \mathrm{Rep}_f(G)\}$. Then
$$m_f(\mathbf{G}_{ad}(R)) \geq h_f(\Phi, p, n),$$
where $h_f(\Phi, p, n)$ is given in the following table

Φ		$h_f(\Phi, p, n)$
A_1	$p \geq 3$	$\frac{1}{2}(p^n - p^{n-1})$
A_ℓ	$\ell \geq 2,\ p \geq 3$	$(p^n - p^{n-1})\, p^{(\ell-1)n}$
B_ℓ	$\ell \geq 3,\ p \geq 3$	$(p^n - p^{n-1})\, p^{(2\ell-3)n}$
C_ℓ	$\ell \geq 2,\ p \geq 3$	$\frac{1}{2}(p^n - p^{n-1})\, p^{(\ell-1)n}$
D_ℓ	$\ell \geq 4,\ p \geq 3$	$(p^n - p^{n-1})\, p^{(2\ell-4)n}$
G_2	$p \geq 5$	$(p^n - p^{n-1})\, p^{2n}$
F_4	$p \geq 3$	$(p^n - p^{n-1})\, p^{7n}$
E_6	$p \geq 3$	$(p^n - p^{n-1})\, p^{10n}$
E_7	$p \geq 3$	$(p^n - p^{n-1})\, p^{16n}$
E_8	$p \geq 3$	$(p^n - p^{n-1})\, p^{28n}$

References

- M. Bardestani, C. Karimianpour, K. Mallahi-Karai and H. Salmasian. *Minimal degree of faithful representations of Chevalley groups over $\mathbb{Z}/p^n\mathbb{Z}$*,, submitted to Journal of Algebra, ArXiv: 1403.3722, (2014).

STUDC CMS SMC

CMS SMC

uOttawa