Let $p$ be a prime, let $S$ be a finite set of primes $q \equiv 1 \mod p$ but $q \not\equiv 1 \mod p^2$ and let $G_S$ be the Galois group of the maximal $p$-extension of $\mathbb{Q}$ unramified outside of $S$. If $\rho$ is a continuous homomorphism of $G_S$ into $\text{GL}_2(\mathbb{Z}_p)$ we use the Koch presentation of $G_S$ and the theory of mild pro-$p$-groups to show that if $p > 3$ then, under certain conditions on the linking numbers of the primes in $S$, either $\rho = 1$ or $\rho(G_S)$ is a Sylow $p$-subgroup of $\text{SL}_2(\mathbb{Z}_p)$. Under certain conditions on $S$ with $|S| = 2, 3$, we show that $\rho = 1$. 