Ordered \( k \)-colorings dichotomy

We introduce three variants of proper colorings with imposed partial ordering on the set of colors and will present dichotomy theorems that separate these problems into tractable and \( NP \)-complete.

Vertices of all considered graphs are integers from 1 to \(|V(G)|\), hence they form a linearly ordered set \((V(G), \leq)\). The set of vertices colored \( c \) will be denoted by \( V_c \). Given a partially ordered set (poset) \((\mathcal{C}, \preceq)\) of colors, in the first problem we want to (properly) color vertices of \( G \) by colors in \( \mathcal{C} \) (color \( G \) by poset \( \mathcal{C} \)) such that for any two colors \( c \) and \( c' \) if \( c \preceq c' \) then for any two vertices \( u \in V_c \) and \( v \in V_{c'} \), \( u \leq v \). Thus, if \( \preceq \) is the empty relation on \( \mathcal{C} \), then the problem is whether \( G \) can be properly colored with \(|\mathcal{C}|\) colors, a well known graph coloring problem.

In the second problem, we want to color \( G \) by poset \( \mathcal{C} \) such that for any two colors \( c \) and \( c' \) if \( c \preceq c' \) then for any two adjacent vertices \( u \in V_c \) and \( v \in V_{c'} \), \( u \leq v \). This problem is the well-known directed graph homomorphism problem whose dichotomy was extensively studied.

In the last problem, we want to color \( G \) by poset \( \mathcal{C} \) such that for any two colors \( c \) and \( c' \) if \( c \preceq c' \) then for any two vertices \( u \in V_c \) and \( v \in V_{c'} \) in a component induced by \( V_c \cup V_{c'} \), \( u \leq v \).

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