Numerical Analysis for Hyperbolic Systems
Analyse numérique des systèmes hyperboliques

(Org: **Paul Arminjon** (Montréal), **Marc Laforest** (Ecole Polytechnique de Montréal) and/et **Emmanuel Lorin** (UOIT))

PAUL ARMINJON, Université de Montréal

Central Finite Volume Methods for 3-dimensional Magnetohydrodynamics

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BRUNO DESPRÈS, CEA

Stability of high order transport schemes in L^1 and L^∞

Computing the solution of the transport equation

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+a\frac{u_{j+\frac{1}{2}}^n-u_{j-\frac{1}{2}}^n}{\Delta x}=0.$$

is a fundamental tool in the numerical solution of many hyperbolic problems. We are interested in the numerical analysis of some very high order Finite Volumes explicit schemes recently discussed in the literature.

We shall explain why *all odd order linear explicit schemes derived form the upwind scheme are asymptotically stable in L^1 and L^∞*, that is

$$\|u^n\|_\infty \le K\|u^0\|_\infty \quad \text{and} \quad \|u^n\|_1 \le K\|u^0\|_1, \qquad \forall n.$$

This result is a way to bypass the standard obstruction result of Godunov about the nonexistence of high order linear schemes with the maximum principle.

We shall discuss some consequences on cartesian grids of this result for hydrodynamic problems and for the 3D wave equation with nonconstant coefficients. Moreover all the schemes we consider are stable with CFL=1 or 2.

JEAN-MICHEL GHIDAGLIA, ENS Cachan and CNRS, CMLA, 61 av. du Pdt Wilson, 94235 Cachan Cedex, France

On the simulation of aerated flows

One of the challenges in Computational Fluid Dynamics (CFD) is to determine efforts exerted by waves on structures, especially coastal structures. The flows associated with wave impact can be quite complicated. In particular, wave breaking can lead to flows that cannot be described by usual models like, e.g., the free-surface Euler or Navier–Stokes equations.

In a free-surface model, the boundary between the gas (air) and the liquid (water) is a surface. The liquid flow is assumed to be incompressible, while the gas is represented by a media, above the liquid, in which the pressure is constant (the atmospheric pressure in general). Such a description is known to be valid for calculating the propagation in the open sea of waves with moderate amplitude, which do not break. Clearly it is not satisfactory when waves either break or hit coastal structures like offshore platforms, jetties, piers, breakwaters, etc.

Our goal is to investigate a relatively simple two-fluid model that can handle breaking waves. It belongs to the family of averaged models and reads as follows:

$$(\alpha^+\rho^+)_t+\operatorname{div}(\alpha^+\rho^+\mathrm{u})=0, \tag{1}$$

$$(\alpha^-\rho^-)_t+\operatorname{div}(\alpha^-\rho^-\mathrm{u})=0, \tag{2}$$

$$(\rho\mathrm{u})_t+\operatorname{div}(\rho\mathrm{u}\otimes\mathrm{u}+p\mathrm{I})=\rho\mathrm{g}, \tag{3}$$

$$(\rho E)_t+\operatorname{div}(\rho H\mathrm{u})=\rho\mathrm{g}\cdot\mathrm{u}, \tag{4}$$

where the superscripts $\pm$ are used to denote liquid and gas respectively. In this model we show that the pressure p is given as a function of three parameters, namely $\alpha \equiv \alpha^{+} - \alpha^{-}$, ρ and e:

$$p = \mathcal{P}(\alpha, \rho, e). \tag{5}$$

PAULINE GODILLON-LAFITTE, Universite Lille 1, Cité Scientifique, Villeneuve d'Ascq
Simulations numériques pour un modèle de pollution atmosphérique

On s'intéresse à la simulation numérique de deux régimes différents pour des particules de polluants interagissant avec l'air. Le fluide, compressible, est décrit par les équations d'Euler et les particules par une equation de type Fokker–Planck, et le système total est couplé par des forces de friction. On étudie plus précisement le cas de la gravité comme force extérieure dans le cas monodimensionnel. Différentes échelles apparaissant naturellement dans la modélisation, on s'attache à ce que les schémas soient encore asymptotiquement valides.

BARBARA KEYFITZ, Fields Institute and University of Houston
Shocks, Rarefactions and Triple Points in Multidimensional Conservation Laws

For the past few years, our research team (Canic, Chern, Jegdic, Kim, Lieberman and Keyfitz) has been working on a self-similar approach to the analysis of systems of conservation laws in two space dimensions. Following work of Tesdall and Hunter [SIAP, 2003] which found a new shock reflection pattern in the unsteady transonic small disturbance equations, we (Tesdall, Sanders and Keyfitz) have now exhibited this Guderley Mach reflection in numerical simulations of a number of systems—specifically the nonlinear wave system and the Eulerian gas dynamics equations for compressible adiabatic flow in two space dimensions. Within this complicated pattern, the details of how the rarefaction wave interacts with the sonic line form a mathematically appealing subproblem. We present some numerical and analytical results on this problem.

BOUALEM KHOUIDER, University of Victoria. PO Box 3045 STN CSC, Victoria, BC, Canada V8W 3P4
Well balanced numerical schemes for the equatorial wave guide

Because of the vanishing Coriolis force at the equator. This latter acts as a waveguide for a large spectrum of waves that are trapped in its vicinity and propagate in the zonal (east-west) direction. The so-called equatorially trapped waves are observed to play a key role in the large-scale organization of convection and other storms in the tropics. They include both dispersive and non-dispersive waves, which interact nonlinearly with each other, with the small scale convective processes, and with the planetary-barotropic Rossby waves. This latter mechanism is believed to be key for tropical and extra-tropical energy-exchanges; means by which the midlatitude weather is influenced by tropical climate-variability. In this talk, we shall discuss some simple-idealized models for the tropical climate and waves using state-of-the-art well-balanced numerical techniques to capture some of balanced dynamics (between the Coriolis force and the meridional gradient of pressure) and various nonlinear wave-interactions.

LILIA KRIVODONOVA, University of Waterloo, Waterloo, ON, Canada
High-Order Discontinuous Galerkin Method for Problems with Shocks

Solutions of nonlinear systems of conservation laws often contain both discontinuities and rich smooth structures. Resolving these simultaneously might be difficult. Discontinuous Galerkin methods are a promising approach to high resolution computations of compressible flows with shocks in general domains. However, solution or flux limiting strategies are needed to restrict or suppress oscillations near discontinuities. Unfortunately, such limiters frequently identify regions near smooth extrema as requiring limiting and this typically results in a reduction of the optimal high-order convergence rate.

We present a slope limiter for discontinuous Galerkin solutions of hyperbolic conservation laws designed to work with an arbitrary-order spatial approximation. It is problem independent and parameter free. The limiter limits not only the solution, but its derivatives as well, which is done adaptively. As a result, limiting of smooth extrema is avoided for quadratic and higher approximations. We show numerically that the $(p+1)$-st rate of convergence can be achieved in smooth regions, while stability is maintained near shocks. Two-dimensional examples on structured meshes will be presented.

FRÉDÉRIC LAGOUTIÈRE, Université Paris-Diderot, 175 rue du Chevaleret, 75013 Paris, France
Analysis of the upwind scheme with probabilities

We provide a probabilistic analysis of the upwind scheme for d-dimensional transport equations on general meshes. One of the purposes of this analysis is to furnish a new "simple" proof of the $1/2$ convergence order of the upwind scheme for non-smooth initial data. The analysis relies on a new interpretation of the scheme, as the *expectation of a random scheme*. We prove that the numerical solution is the expectation of the initial data on the foot of a random characteristic (instead of the initial data on the foot of the exact characteristic of the transport problem). Then the general idea of the analysis is to prove

- first, that the random characteristics are driven *in mean* by the exact ones,
- second, that the fluctuations of the random characteristics around these exact characteristics are of order $Ch^{1/2}$ where h is the maximal cell diameter in the mesh and C only depends on the initial datum and the time: this means that the random characteristics are of *diffusive* type.

This is done *via* Central Limit type Theorems, or, more precisely, with martingale estimates.

We finally prove the $1/2$ order in $L^\infty\big([0,T],L^1(\mathbb{R}^d)\big)$ for BV initial data, and the $1/2-\varepsilon$ rate in $L^\infty\big([0,T],L^\infty(\mathbb{R}^d)\big)$ for Lipschitz-continuous initial data (for any $\varepsilon>0$).

Besides, this analysis provides a new explanation of the well-known *dissipative* behavior of the upwind scheme, by means of stochastic processes (in the same way as the Brownian motion for the heat equation).

NATHALIE LANSON, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1
Convergence Analysis of Renormalized Meshfree Schemes

Meshfree methods, also referred to as particle methods, have been recently developed for the approximation of hyperbolic problems and are now used in a wide range of applications, due to their ability to handle complex situations involving highly distorted systems. Renormalized meshfree schemes are based on a new class of approximation of derivatives that allows for better accuracy than classical particle methods. In this talk, I will discuss the analytical aspect of renormalized meshfree schemes; stability results will be presented as well as the geometrical conditions insuring stability, and the convergence of the schemes in the case of nonlinear scalar conservation laws will be established. Finally, the analogy made between finite volume schemes and meshfree schemes within the analysis will lead to the construction of some hybrid schemes.

PHILIPPE G. LEFLOCH, University of Paris 6
Hyperbolic conservation laws on manifolds: well-posedness theory and numerical approximation

Kruzkov's theory of discontinuous solutions to nonlinear hyperbolic conservation laws in several space dimensions is restricted to the (flat) Euclidian space. In this lecture, motivated by applications to geophysical fluid flows modeled by the shallow water equations, I will present the foundations for theoretical and numerical studies of entropy solutions to hyperbolic equations posed on a curved manifold. The aim of this research is to investigate some interplay between the manifold's geometry and the behavior of discontinuous solutions to partial differential equations.

In this lecture, I will discuss the well-posedness theory, the derivation of the $L1$ contraction property, the convergence of the finite volume schemes, the $L1$ error estimate, and the practical implementation in the case of the sphere.

FRÉDÉRIC PASCAL, CMLA, ENS de Cachan, 61 avenue du Pt Wilson, 94235 Cachan, France
On the supra-convergence phenomenon of the cell-centered finite volume method

The contribution investigates the supra-convergence phenomenon which occurs in finite volume methods used to approximate hyperbolic problems on a bounded domain. These methods which take into account the direction where the information comes from are well-adapted for the discretization of such problems. However, even for simple model problems, the theoretical analysis of the error estimate is still a challenging task. One of the main difficulties holds in the fact that the non-uniformity of unstructured meshes brings up an apparent loss of consistency, at least in the finite differences sense. This loss due to the upwind part of the scheme is an artifact of standard convergence proof based on the Lax–Richtmyer theorem. Actually, the scheme maintains the accuracy, the global error behaves in better way than the local error does and converges to zero with the parameter of discretization. This property of enhancement of the truncation error is called supra-convergence.

In order to tackle this lack of consistency, we proceed, for the mathematical analysis, by correcting the error with a geometric corrector introduced for the linear convection problem with constant velocity vector. We first describe the principle of this mathematical analysis. Then we discuss an extension of the notion of geometric corrector to the non-constant velocity case in one dimension since, with the difference of dimension two, an explicit formula of the geometric corrector is available. For a nonlinear conservation law, we are also able to adapt the formula and we can prove that, as long time as the solution remains smooth, the scheme is first order accurate.

DOMINIK SCHÖTZAU, University of British Columbia
Energy norm a-posteriori error estimation for hp-adaptive DG methods for convection-diffusion equations

We develop the energy norm a-posteriori error estimation of hp-adaptive discontinuous Galerkin (DG) finite element methods for stationary convection-diffusion equations. In particular, we derive computable upper and lower bounds on the error measured in terms of a natural (mesh-dependent) energy norm and a dual norm associated with the convective terms in the equations. The bounds are fully explicit in the local mesh sizes and approximation orders. The ratio of the upper and lower bounds is independent of the magnitude of the Péclet number of the problem, and hence the estimator is robust for convection-dominated problems.

Our theoretical findings are illustrated in a series of numerical experiments.

MARIE-ODETTE ST-HILAIRE, Université de Montréal
Toward an improved capture of stiff detonation waves

The simultaneous presence of two scales: macroscopic for the gas flow and microscopic for the chemical reaction, makes numerical approximation of detonation waves very delicate. A resolved simulation, where the small chemical time scale is fully resolved, effectively captures the wave in details. However, it is far too expensive in computing time, especially for multi-dimensional problems. While being economic, an underresolved approach, where the discretisation is proportional to the macroscopic scale, is unfortunately inefficient for the capture of stiff detonation waves because it leads to unphysical solutions.

We propose a family of accurate time-splitting methods, numerically stable, allowing underresolved calculations and requiring neither the resolution of the Riemann problem nor the knowledge of the characteristic structure of the flux jacobian matrix and of course, converging to the physical solution. With a refinement of the grid, these methods moreover effectively capture the unstable character of the detonation and provide the exact front structure of the wave. It is realistic to claim that such methods can moreover solve about any hyperbolic system with source term. We thus elaborate "black box"-type methods, while the majority of the schemes existing for the detonation problem use properties of the solution.