PROBLEMS FOR MAY

Please send your solutions to

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individually as you solve the problems. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

675. ABC is a triangle with circumcentre O such that $\angle A$ exceeds 90° and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

676. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

for all real x and y.

677. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a} \times (\mathbf{b} - \mathbf{c})]^2 + [\mathbf{b} \times (\mathbf{c} - \mathbf{a})]^2 + [\mathbf{c} \times (\mathbf{a} - \mathbf{b})]^2 = (\mathbf{b} \times \mathbf{c})^2 + (\mathbf{c} \times \mathbf{a})^2 + (\mathbf{a} \times \mathbf{b})^2 + (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})^2 \ .$$

678. For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc} \ .$$

679. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

680. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} \ .$$

681. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbb{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

682. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

683. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$