Problems

1. Find all integer solutions to the equation $7x^2y^2 + 4x^2 = 77y^2 + 1260$.

2. A polynomial $f(x)$ with integer coefficients is said to be *tri-divisible* if 3 divides $f(k)$ for any integer k. Determine necessary and sufficient conditions for a polynomial to be tri-divisible.

3. Let N be a 3-digit number with three distinct non-zero digits. We say that N is *mediocre* if it has the property that when all six 3-digit permutations of N are written down, the average is N. For example, $N = 481$ is mediocre, since it is the average of $\{418, 481, 148, 184, 814, 841\}$.

   Determine the largest mediocre number.

4. Given an acute-angled triangle ABC whose altitudes from B and C intersect at H, let P be any point on side BC and X, Y be points on AB, AC, respectively, such that $PB = PX$ and $PC = PY$. Prove that the points A, H, X, Y lie on a common circle.

5. Let x and y be positive real numbers such that $x + y = 1$. Show that

$$\left(\frac{x+1}{x}\right)^2 + \left(\frac{y+1}{y}\right)^2 \geq 18.$$

6. Let $\triangle ABC$ be a right-angled triangle with $\angle A = 90°$, and $AB < AC$. Let points D, E, F be located on side BC so that AD is the altitude, AE is the internal angle bisector, and AF is the median.

   Prove that $3AD + AF > 4AE$.

7. A $(0_x, 1_y, 2_z)$-string is an infinite ternary string such that:

   - If there is a 0 in position i, then there is a 1 in position $i + x$
   - If there is a 1 in position j then there is a 2 in position $j + y$,
   - if there is a 2 in position k then there is a 0 in position $k + z$.

   For how many ordered triples of positive integers (x, y, z) with $x, y, z \leq 100$ does there exist $(0_x, 1_y, 2_z)$-string?

8. A magical castle has n identical rooms, each of which contains k doors arranged in a line. In room i, $1 \leq i \leq n - 1$ there is one door that will take you to room $i + 1$, and in room n there is one door that takes you out of the castle. All other doors take you back to room 1. When you go through a door and enter a room, you are unable to tell what room you are entering and you are unable to see which doors you have gone through before. You begin by standing in room 1 and know the values of $nandk$. Determine for which values of n and k there exists a strategy that is guaranteed to get you out of the castle and explain the strategy. For such values of n and k, exhibit such a strategy and prove that it will work.