Problems

1. Find all integer solutions to the equation $7x^2y^2 + 4x^2 = 77y^2 + 1260$.

2. A polynomial $f(x)$ with integer coefficients is said to be tri-divisible if 3 divides $f(k)$ for any integer $k$. Determine necessary and sufficient conditions for a polynomial to be tri-divisible.

3. Let $N$ be a 3-digit number with three distinct non-zero digits. We say that $N$ is mediocre if it has the property that when all six 3-digit permutations of $N$ are written down, the average is $N$. For example, $N = 481$ is mediocre, since it is the average of \{418, 481, 148, 184, 814, 841\}. Determine the largest mediocre number.

4. Given an acute-angled triangle $ABC$ whose altitudes from $B$ and $C$ intersect at $H$, let $P$ be any point on side $BC$ and $X, Y$ be points on $AB, AC$, respectively, such that $PB = PX$ and $PC = PY$. Prove that the points $A, H, X, Y$ lie on a common circle.

5. Let $x$ and $y$ be positive real numbers such that $x + y = 1$. Show that

$$\left(\frac{x+1}{x}\right)^2 + \left(\frac{y+1}{y}\right)^2 \geq 18.$$ 

6. Let $\triangle ABC$ be a right-angled triangle with $\angle A = 90^\circ$, and $AB < AC$. Let points $D, E, F$ be located on side $BC$ so that $AD$ is the altitude, $AE$ is the internal angle bisector, and $AF$ is the median. Prove that $3AD + AF > 4AE$.

7. A $(0_x, 1_y, 2_z)$-string is an infinite ternary string such that:

- If there is a 0 in position $i$, then there is a 1 in position $i + x$
- If there is a 1 in position $j$ then there is a 2 in position $j + y$,
- if there is a 2 in position $k$ then there is a 0 in position $k + z$.

For how many ordered triples of positive integers $(x, y, z)$ with $x, y, z \leq 100$ does there exist $(0_x, 1_y, 2_z)$-string?

8. A magical castle has $n$ identical rooms, each of which contains $k$ doors arranged in a line. In room $i$, $1 \leq i \leq n - 1$ there is one door that will take you to room $i + 1$, and in room $n$ there is one door that takes you out of the castle. All other doors take you back to room 1. When you go through a door and enter a room, you are unable to tell what room you are entering and you are unable to see which doors you have gone through before. You begin by standing in room 1 and know the values of $n$ and $k$. Determine for which values of $n$ and $k$ there exists a strategy that is guaranteed to get you out of the castle and explain the strategy. For such values of $n$ and $k$, exhibit such a strategy and prove that it will work.