Sun Life Financial

2012 Sun Life Financial Repêchage Competition

1. The front row of a movie theatre contains 45 seats.

   (a) If 42 people are sitting in the front row, prove that there are 10 consecutive seats that are all occupied.

   (b) Show that this conclusion doesn't necessarily hold if only 41 people are sitting in the front row.

2. Given a positive integer m, let $d(m)$ be the number of positive divisors of m. Determine all positive integers n such that $d(n) + d(n+1) = 5$.

3. We say that (a, b, c) form a *fantastic triplet* if a, b, c are positive integers, a, b, c form a geometric sequence, and $a, b+1, c$ form an arithmetic sequence. For example, $(2, 4, 8)$ and $(8, 12, 18)$ are fantastic triplets. Prove that there exist infinitely many fantastic triplets.

4. Let ABC be a triangle such that $\angle BAC = 90^\circ$ and $AB < AC$. We divide the interior of the triangle into the following six regions:

$$\begin{aligned}
S_1 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PA < PB < PC \\
S_2 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PA < PC < PB \\
S_3 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PB < PA < PC \\
S_4 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PB < PC < PA \\
S_5 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PC < PA < PB \\
S_6 &= \text{set of all points } P \text{ inside } \Delta ABC \text{ such that } PC < PB < PA.
\end{aligned}$$

   Suppose that the ratio of the area of the largest region to the area of the smallest non-empty region is $49 : 1$. Determine the ratio $AC : AB$.

5. Given a positive integer n, let $d(n)$ be the largest positive divisor of n less than n. For example, $d(8) = 4$ and $d(13) = 1$. A sequence of positive integers $a_1, a_2, \ldots$ satisfies

$$a_{i+1} = a_i + d(a_i),$$

   for all positive integers i. Prove that regardless of the choice of a_1, there are infinitely many terms in the sequence divisible by 3^{2011}.

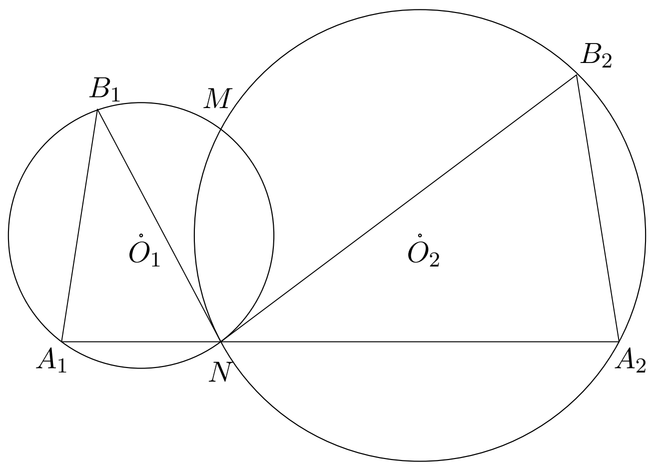
6. Determine whether there exist two real numbers a and b such that both $(x-a)^3 + (x-b)^2 + x$ and $(x-b)^3 + (x-a)^2 + x$ contain only real roots.

7. Six tennis players gather to play in a tournament where each pair of persons play one game, with one person declared the winner and the other person the loser. A triplet of three players $\{A, B, C\}$ is said to be *cyclic* if A wins against B, B wins against C and C wins against A.

   (a) After the tournament, the six people are to be separated in two rooms such that none of the two rooms contains a cyclic triplet. Prove that this is always possible.

(b) Suppose there are instead seven people in the tournament. Is it always possible that the seven people can be separated in two rooms such that none of the two rooms contains a cyclic triplet?

8. Suppose circles W_1 and W_2, with centres O_1 and O_2 respectively, intersect at points M and N. Let the tangent on W_2 at point N intersect W_1 for the second time at B_1. Similarly, let the tangent on W_1 at point N intersect W_2 for the second time at B_2. Let A_1 be a point on W_1 which is on arc B_1N not containing M and suppose line A_1N intersects W_2 at point A_2. Denote the incentres of triangles B_1A_1N and B_2A_2N by I_1 and I_2, respectively.[1]



Show that

$$\angle I_1MI_2 = \angle O_1MO_2.$$

[1] Given a triangle ABC, the incentre of the triangle is defined to be the intersection of the angle bisectors of A, B and C. To avoid cluttering, the incentre is omitted in the provided diagram. Note also that the diagram serves only as an aid and is not necessarily drawn to scale.