2009 Canadian Mathematical Olympiad Qualification Repêchage

Instructions

- Answer as many problems as you can, presenting a full solution, neatly written, to each problem that you answer.
- Each solution should be clearly labelled with the problem number and should include all steps necessary to solve the problem.
- You should solve the problems on your own without outside help.
- Submit your finished solutions on or before Monday, January 12, 2009. There are two options for submitting your solutions:
  - Type your solutions into a single document (not multiple documents), and send by email to mathteachers@math.uwaterloo.ca.
    Clearly label any email submission as "2009 CMO Qualification Repêchage".
  - Handwrite your solutions and send by courier or regular mail (preferably by courier) to

    2009 CMO Qualification Repêchage
    Centre for Education in Mathematics and Computing
    Faculty of Mathematics, Room MC 5104
    University of Waterloo
    200 University Ave. W.
    Waterloo, ON N2L 3G1
    CANADA

Problems

1. Determine all solutions to the system of equations

$$\begin{aligned} x+y+z &= 2 \\ x^2-y^2-z^2 &= 2 \\ x-3y^2+z &= 0 \end{aligned}$$

2. Triangle ABC is right-angled at C with $AC = b$ and $BC = a$. If d is the length of the altitude from C to AB, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{d^2}$.

3. Prove that there does not exist a polynomial $f(x)$ with integer coefficients for which $f(2008) = 0$ and $f(2010) = 1867$.

4. Three fair six-sided dice are thrown. Determine the probability that the sum of the numbers on the three top faces is 6.

5. Determine all positive integers n for which $n(n + 9)$ is a perfect square.

6. Triangle ABC is right-angled at C. AQ is drawn parallel to BC with Q and B on opposite sides of AC so that when BQ is drawn, intersecting AC at P, we have $PQ = 2AB$. Prove that $\angle ABC = 3\angle PBC$.

7. A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

8. Determine an infinite family of quadruples (a, b, c, d) of positive integers, each of which is a solution to $a^4 + b^5 + c^6 = d^7$.

9. Suppose that m and k are positive integers. Determine the number of sequences $x_1, x_2, x_3, \ldots, x_{m-1}, x_m$ with

   - x_i an integer for $i = 1, 2, 3, \ldots, m$,
   - $1 \leq x_i \leq k$ for $i = 1, 2, 3, \ldots, m$,
   - $x_1 \neq x_m$, and
   - no two consecutive terms equal.

10. Ten boxes are arranged in a circle. Each box initially contains a positive number of golf balls. A move consists of taking all of the golf balls from one of the boxes and placing them into the boxes that follow it in a counterclockwise direction, putting one ball into each box. Prove that if the next move always starts with the box where the last ball of the previous move was placed, then after some number of moves, we get back to the initial distribution of golf balls in the boxes.