Canadian Mathematical Olympiad
1969

Problem 1

Show that if $a_1/b_1 = a_2/b_2 = a_3/b_3$ and p_1, p_2, p_3 are not all zero, then

$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1 a_1^n + p_2 a_2^n + p_3 a_3^n}{p_1 b_1^n + p_2 b_2^n + p_3 b_3^n}$$

for every positive integer n.

Problem 2

Determine which of the two numbers $\sqrt{c+1} - \sqrt{c}$, $\sqrt{c} - \sqrt{c-1}$ is greater for any $c \geq 1$.

Problem 3

Let c be the length of the hypotenuse of a right angle triangle whose other two sides have lengths a and b. Prove that $a + b \leq \sqrt{2}c$. When does the equality hold?

Problem 4

Let ABC be an equilateral triangle, and P be an arbitrary point within the triangle. Perpendiculars PD, PE, PF are drawn to the three sides of the triangle. Show that, no matter where P is chosen,

$$\frac{PD + PE + PF}{AB + BC + CA} = \frac{1}{2\sqrt{3}}.$$

Problem 5

Let ABC be a triangle with sides of lengths a, b and c. Let the bisector of the angle C cut AB in D. Prove that the length of CD is

$$\frac{2ab\cos\frac{C}{2}}{a+b}.$$

Problem 6

Find the sum of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (n-1)(n-1)! + n \cdot n!$, where $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

Problem 7

Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

Problem 8

Let f be a function with the following properties:
1) $f(n)$ is defined for every positive integer n;
2) $f(n)$ is an integer;
3) $f(2) = 2$;
4) $f(mn) = f(m)f(n)$ for all m and n;
5) $f(m) > f(n)$ whenever $m > n$.

Prove that $f(n) = n$.

Problem 9

Show that for any quadrilateral inscribed in a circle of radius 1, the length of the shortest side is less than or equal to $\sqrt{2}$.

Problem 10

Let ABC be the right-angled isosceles triangle whose equal sides have length 1. P is a point on the hypotenuse, and the feet of the perpendiculars from P to the other sides are Q and R. Consider the areas of the triangles APQ and PBR, and the area of the rectangle $QCRP$. Prove that regardless of how P is chosen, the largest of these three areas is at least $2/9$.