PROBLEMS FOR MARCH

Please send your solutions to

E.J. Barbeau
Department of Mathematics
University of Toronto
40 St. George Street
Toronto, ON M5S 2E4

individually as you solve the problems. Electronic files can be sent to barbeau@math.utoronto.ca. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

668. The nonisosceles right triangle $ABC$ has $\angle CAB = 90^\circ$. The inscribed circle with centre $T$ touches the sides $AB$ and $AC$ at $U$ and $V$ respectively. The tangent through $A$ of the circumscribed circle meets $UV$ produced in $S$. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where $r$ is the radius of the inscribed circle and $d_1$ and $d_2$ are the respective distances from $S$ to $AC$ and $AB$.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^n} - n^n,$$

i.e., the number on the right is a multiple of 1989.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \ldots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with “carrying” occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ The real number $\tau$ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer $n$, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer $n$, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let $G_n$ be any one of the functions $F_{n+1} F_n$, $F_{n+1} F_{n-1}$ and $F_n^2$. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. $ABC$ is an isosceles triangle with $AB = AC$. Let $D$ be the point on the side $AC$ for which $CD = 2AD$. Let $P$ be the point on the segment $BD$ such that $\angle APC = 90^\circ$. Prove that $\angle ABP = \angle PCB$.

674. The sides $BC$, $CA$, $AB$ of triangle $ABC$ are produced to the points $R$, $P$, $Q$ respectively, so that $CR = AP = BQ$. Prove that triangle $PQR$ is equilateral if and only if triangle $ABC$ is equilateral.