Solutions for November

647. Find all continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(x + f(y)) = f(x) + y
\]

for every \( x, y \in \mathbb{R} \).

Solution 1. Setting \( (x, y) = (t, 0) \) yields \( f(t + f(0)) = f(t) \) for all real \( t \). Setting \( (x, y) = (0, t) \) yields \( f(f(t)) = f(0) + t \) for all real \( t \). Hence \( f(f(t)) = f(t) \) for all real \( t \), i.e., \( f(f(z)) = z \) for each \( z \) in the image of \( f \). Let \( (x, y) = (f(t), -f(0)) \). Then

\[
f(f(t) + f(-f(0))) = f(f(t)) - f(0) = f(0) + t - f(0) = t
\]

so that the image of \( f \) contains every real and so \( f(f(t)) \equiv t \) for all real \( t \).

Taking \( (x, y) = (u, f(v)) \) yields

\[
f(u + v) = f(u) + f(v)
\]

since \( v = f(f(v)) \) for all real \( u \) and \( v \). In particular, \( f(0) = 2f(0) \), so \( f(0) = 0 \) and \( 0 = f(-t + t) = f(-t) + f(t) \). By induction, it can be shown that for each integer \( n \) and each real \( t \), \( f(nt) = nf(t) \). In particular, for each rational \( r/s \), \( f(r/s) = rf(1/s) = (r/s)f(1) \). Since \( f \) is continuous, \( f(t) = f(t \cdot 1) = tf(1) \) for all real \( t \). Let \( c = f(1) \). Then \( 1 = f(1) = f(c) = cf(1) = c2 \) so that \( c = \pm 1 \). Hence \( f(t) = t \) or \( f(t) = -t \). Checking reveals that both these solutions work. (For \( f(t) \equiv -t \), \( f(x + f(y)) = -x - f(y) = f(x) + y \), as required.)

Solution 2. Taking \( (x, y) = (0, 0) \) yields \( f(f(0)) = f(0) \), whence \( f(f(f(0))) = f(f(0)) = f(0) \). Taking \( (x, y) = (0, f(0)) \) yields \( f(f(f(0))) = 2f(0) \). Hence \( 2f(0) = f(0) \) so that \( f(0) = 0 \). Taking \( x = 0 \) yields \( f(f(y)) = y \) for each \( y \). We can complete the solution as in the Second Solution.

Solution 3. [J. Rickards] Let \( (x, y) = (x, -f(x)) \) to get

\[
f(x + f(-f(x)) = f(x) - f(x) = 0
\]

for all \( x \). Thus, there is at least one element \( u \) for which \( f(u) = 0 \). But then, taking \( (x, y) = (0, u) \), we find that \( f(0) = f(0 + f(u)) = f(0) + u \), so that \( u = 0 \).

Therefore \( f(f(y)) = y \) for each \( y \), so that \( f \) is a one-one onto function. Also, \( x + f(-f(x)) = 0 \), so that \( -f(x) = f(f(-f(x))) = f(-x) \) for each value of \( x \).

Since \( f(x) \) is continuous and vanishes only for \( x = 0 \), we have either (1) \( f(x) \) is positive for \( x > 0 \) and negative for \( x < 0 \), or (2) \( f(x) \) is negative for \( x > 0 \) and positive for \( x < 0 \). Suppose that situation (1) obtains. Then, for every real number \( x \), \( f(x - f(x)) = f(x + f(-x)) = f(x) - x = -(x - f(x)) \). Since \( f(x - f(x)) \) and \( x - f(x) \) have the same sign, we must have \( f(x) = x \). Suppose that situation (2) obtains. Then, for every real \( x \), \( f(x + f(x)) = f(x) + x \), from which we deduce that \( f(x) = -x \). Therefore, there are two functions \( f(x) = x \) and \( f(x) = -x \) that satisfy the equation and both work.

648. Prove that for every positive integer \( n \), the integer \( 1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n} \) is composite.

Solution. Observe the following representations:

\[
x8 + x6 + x4 + x2 + 1 = (x4 + x3 + x2 + x + 1)(x4 - x3 + x2 - x + 1)
\]

(1)

and

\[
x4 + x3 + x2 + x + 1 = (x2 + 3x + 1)2 - 5x(x + 1)2
\]

(2)

When \( n = 2k \) is even, we can substitute \( x = 5^k \) into equation (1) to get a factorization. When \( n = 2k - 1 \) is odd, we can substitute \( x = 5^{2k-1} \) into equation (2) to get a difference of squares, which can then be factored.
In the triangle \(ABC\), \(\angle BAC = 20^\circ\) and \(\angle ACB = 30^\circ\). The point \(M\) is located in the interior of triangle \(ABC\) so that \(\angle MAC = \angle MCA = 10^\circ\). Determine \(\angle BMC\).

**Solution 1.** [S. Sun] Construct equilateral triangle \(MDC\) with \(M\) and \(D\) on opposite sides of \(AC\) and equilateral triangle \(AME\) with \(M\) and \(Z\) on opposite sides of \(AB\). Since \(AM = MD\), these equilateral triangles are congruent. Since \(AM = MD\) and \(\angle MAD = \angle MDA = 40^\circ\). Since \(ME = AM = MC\), triangle \(EMC\) is isosceles. Since \(\angle EMC = \angle MCE = 20^\circ\). As \(\angle MCB = 20^\circ = \angle MCE\), \(E, B, C\) are collinear. Now \(\angle EBA = \angle BAC + \angle BCA = 20^\circ + 30^\circ = 50^\circ\) = \(60^\circ - 10^\circ\) = \(\angle EAM - \angle BAM = \angle EAB\), so that \(BE = AE = ME\) and triangle \(BEM\) is isosceles. Since \(\angle BEM = \angle BEA - \angle MEA = 80^\circ - 60^\circ = 20^\circ\), it follows that \(\angle BMC = 360^\circ - \angle EMB - \angle EMA - \angle AMC = 360^\circ - 80^\circ - 60^\circ - 160^\circ = 60^\circ\).

**Solution 2.** Let \(O\) be the circumcentre of the triangle \(BAC\); this lies on the opposite side of \(AC\) to \(B\). Since the angle subtended at the centre by a chord is double that subtended at the circumference, we have that \(\angle AOC = 2(180^\circ - \angle ABC) = 2(180^\circ - 130^\circ) = 100^\circ\).

The right bisector of the segment \(AC\) passes through the apex of the isosceles triangle \(MAC\) and the centre \(O\) of the circumcircle of triangle \(BAC\). We have that \(\angle AOM = 50^\circ\), \(\angle AMO = \frac{1}{2}\angle AMC = 80^\circ\), and \(\angle MAO = 180^\circ - 50^\circ - 80^\circ = 50^\circ\).

Therefore, triangle \(MAO\) is isosceles with \(MA = MO\).

Observe that \(\angle BAO = \angle BAC + \angle MAB - \angle MAC = 60^\circ\) and that \(AO = BO\), so that triangle \(BAO\) is equilateral and so \(BA = BO\). Since \(B\) and \(M\) are both equidistant from \(A\) and \(O\), the line \(BM\) must right bisect the segment \(AO\) at \(N\), say. Therefore, \(\angle MNO = 90^\circ\), so that \(\angle NMO = 40^\circ\). It follows that \(\angle BMC = 180^\circ - \angle CMO - \angle NMO = 180^\circ - 80^\circ - 40^\circ = 60^\circ\).

**Solution 3.** [M. Essaft] Let \(\alpha = \angle MBA\), so that \(\angle MBC = 130^\circ - \alpha\). From the trigonometric version of Ceva’s Theorem, we have that

\[
\sin \alpha \sin 20^\circ \sin 10^\circ = \sin(130^\circ - \alpha) \sin 10^\circ \sin 10^\circ
\]

\[
\Rightarrow 2 \sin \alpha \sin 10^\circ \cos 10^\circ = \sin(130^\circ - \alpha) \sin 10^\circ
\]

\[
\Rightarrow 2 \sin \alpha \cos 10^\circ = \cos(40^\circ - \alpha) = \cos 40^\circ \cos \alpha + \sin 40^\circ \sin \alpha
\]

Dividing both sides by \(\cos 40^\circ \cos \alpha\) yields that

\[
2 \cos \alpha \left( \frac{2 \cos 10^\circ}{\cos 40^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ} \right) = 1.
\]
Therefore
\[
\cot \alpha = \frac{\cos 10^\circ + \cos 10^\circ - \cos 50^\circ}{\cos 40^\circ}
\]
\[
= \frac{\cos 10^\circ + 2 \sin 30^\circ \sin 20^\circ}{\cos 40^\circ}
\]
\[
= \frac{\cos 10^\circ + \sin 20^\circ}{\cos 40^\circ} = \frac{\cos 10^\circ + \cos 70^\circ}{\cos 40^\circ}
\]
\[
= \frac{2 \cos 40^\circ \cos 30^\circ}{\cos 40^\circ} = 2 \cos 30^\circ = \sqrt{3}.
\]

Therefore \(\alpha = 30^\circ\).

650. Suppose that the nonzero real numbers satisfy
\[
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz}.
\]

Determine the minimum value of
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2}.
\]

Solution 1. [W. Fu] Let \(f(x, y, z)\) denote the expression
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2}.
\]

Then
\[
f(x, y, z) - f(x, z, y) = \left(\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2}\right) - \left(\frac{x^4}{x^2 + z^2} + \frac{z^4}{z^2 + y^2} + \frac{y^4}{y^2 + x^2}\right)
\]
\[
= \frac{x^4 - y^4}{x^2 + y^2} + \frac{y^4 - z^4}{y^2 + z^2} + \frac{z^4 - x^4}{z^2 + x^2}
\]
\[
= (x^2 - y^2) + (y^2 - z^2) + (z^2 - x^2) = 0.
\]

Thus, \(f(x, y, z) = f(x, z, y)\) and
\[
f(x, y, z) = \frac{1}{2} (f(x, y, z) + f(x, z, y))
\]
\[
= \frac{1}{2} \left[ \frac{x^4 + y^4}{x^2 + y^2} + \frac{y^4 + z^4}{y^2 + z^2} + \frac{z^4 + x^4}{z^2 + x^2} \right]
\]
\[
= \frac{1}{2} \left[ (x^2 + y^2 - \frac{2x^2y^2}{x^2 + y^2}) + (y^2 + z^2 - \frac{2y^2z^2}{y^2 + z^2}) + (z^2 + x^2 - \frac{2z^2x^2}{z^2 + x^2}) \right]
\]
\[
= (x^2 + y^2 + z^2) - \frac{1}{2} \left( \frac{2x^2y^2}{x^2 + y^2} + \frac{2y^2z^2}{y^2 + z^2} + \frac{2z^2x^2}{z^2 + x^2} \right)
\]

Observe that
\[
x^2 + y^2 + z^2 = \frac{1}{2} [(x^2 + y^2) + (y^2 + z^2) + (z^2 + x^2)] \geq xy + yz + zx = 1
\]
and that \(2x^2y^2 \leq x^4 + y^4\). Hence
\[
f(x, y, z) \geq 1 - \frac{1}{2} \left( \frac{x^4 + y^4}{x^2 + y^2} + \frac{y^4 + z^4}{y^2 + z^2} + \frac{z^4 + x^4}{z^2 + x^2} \right)
\]
\[
= 1 - \frac{1}{2} [f(x, y, z) + f(x, z, y)] = 1 - f(x, y, z),
\]
from which \( f(x, y, z) \geq \frac{1}{2} \). Equality occurs if and only if \( x = y = z = 1/\sqrt{3} \).

**Solution 2.** [S. Sun] From the Arithmetic-Geometric Means Inequality, we have that
\[
\frac{x^4}{x^2 + y^2} + \frac{1}{4}(x^2 + y^2) \geq x^2
\]
with a similar inequality for the other pairs of variables. Adding the three inequalities obtained, we find that
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} + \frac{1}{2}(x^2 + y^2 + z^2) \geq x^2 + y^2 + z^2
\]
from which
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} \geq \frac{1}{2}(x^2 + y^2 + z^2)
\]
with equality if and only if \( x = y = z \). Since \((x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0\), it follows that \( x^2 + y^2 + z^2 \geq xy + yz + zx \). Therefore
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} \geq \frac{1}{2}
\]
with equality if and only if \( x = y = z = 1/\sqrt{3} \).

**Solution 3.** [K. Zhou; G. Ajjanagadde; M. Essafty] Since \((x - y)^2 \geq 0\), etc., we have that \( x^2 + y^2 + z^2 \geq xy + yz + zx \). By the Cauchy-Schwarz Inequality, we have that
\[
\left[ \left( \frac{x^2}{\sqrt{x^2 + y^2}} \right)^2 + \left( \frac{y^2}{\sqrt{y^2 + z^2}} \right)^2 + \left( \frac{z^2}{\sqrt{z^2 + x^2}} \right)^2 \right] \left[ \sqrt{x^2 + y^2} + \sqrt{y^2 + z^2} + \sqrt{z^2 + x^2} \right]^2 \geq (x^2 + y^2 + z^2)
\]
whence
\[
\left( \frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} \right)(x^2 + y^2) + (y^2 + z^2) + (z^2 + x^2) \geq (x^2 + y^2 + z^2)
\]
so that
\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} \geq \frac{x^2 + y^2 + z^2}{2} \geq \frac{xy + yz + zx}{2} = \frac{1}{2}.
\]
Equality occurs when \( x = y = z = 1/\sqrt{3} \).

**Solution 4.** Observe that the given condition is equivalent to \( xy + yz + zx \geq 1 \). Since the expression to be minimized is the same when \( (x, y, z) \) is replaced by \((-x, -y, -z)\) and since two of the variables must have the same sign, we may assume that \( x \) and \( y \) are both positive.

Suppose, first, that \( z > 0 \). Since \( x^2 + y^2 \geq 2xy \), we have that
\[
\frac{x^4}{x^2 + y^2} = x^2 - \frac{x^2y^2}{x^2 + y^2} \geq x^2 - \frac{xy}{2},
\]
with similar inequalities for the other pairs of variables. Therefore, the expression to be minimized is not less than
\[
(x^2 + y^2 + z^2) - \frac{1}{2}(xy + yz + zx) \geq (xy + yz + zx) - \frac{1}{2}(xy + yz + zx) = \frac{1}{2}.
\]
Equality occurs if and only if \( x = y = z = 1/\sqrt{3} \).

Regardless of the signs of the variables, if the largest of \( x^2, y^2, z^2 \) is at least \( 2 \), we show that the expression is not less than \( 1 \). For example, if \( x^2 \geq 2, x^2 \geq y^2 \), we find that
\[
\frac{x^4}{x^2 + y^2} \geq \frac{x^4}{2x^2} = \frac{x^2}{2} \geq 1.
\]
Henceforth, assume that \(x^2, y^2, z^2\) are less than 2 and that \(z < 0\). Then \(xy < 2\). Since \(0 > z = (1 - xy)/(x + y)\), then \(xy > 1\), so that \(x + y \geq 2\sqrt{xy} > 2\). Hence
\[
|z| = \frac{xy - 1}{x + y} \leq \frac{1}{2}.
\]
If \(x > y\), then (because \(xy > 1\)), \(x > 1\), so that
\[
\frac{x^4}{x^2 + y^2} > \frac{x^4}{2x^2} > \frac{1}{2}.
\]
If \(y > z\), then \(y > 1 > |z|\) and
\[
\frac{y^4}{y^2 + z^2} > \frac{y^4}{2y^2} > \frac{1}{2}.
\]
In any case, when \(z < 0\), the quantity to be minimized exceeds 1/2. Therefore, the minimum value is 1/2, achieved when \((x, y, z) = (\frac{3}{2}, \frac{3}{2}, \frac{3}{2})\).

Solution 5. [B. Wu] We first establish a lemma: if \(a, b, u, v\) are positive, then
\[
a^2 \frac{u}{a} + b^2 \frac{v}{b} \geq (a + b)^2 \frac{u}{a + b}
\]
with equality if and only if \(a : u = b : v\). To see this, subtract the right side from the left to get a fraction whose numerator is \((av - bu)^2\).

Applying this to the given expression yields that
\[
\frac{(x^2)^2}{y^2 + z^2} + \frac{(y^2)^2}{z^2 + x^2} + \frac{(z^2)^2}{x^2 + y^2} \geq \frac{(x^2 + y^2 + z^2)^2}{2(x^2 + y^2 + z^2)} = \frac{x^2 + y^2 + z^2}{2}
\]
\[
\geq \frac{xy + yz + zx}{2} = \frac{1}{2}.
\]
Equality occurs if and only if \(x = y = z = 1/\sqrt{3}\).

Solution 6. [M. Essafty] Squaring both sides of the equation \(2x^2 = (x^2 + y^2) + (x^2 - y^2)\) yields that
\[
4x^4 = (x^2 + y^2)^2 + (x^2 - y^2)^2 + 2(x^2 + y^2)(x^2 - y^2) \geq (x^2 + y^2)^2 + 2(x^2 + y^2)(x^2 - y^2)
\]
whence
\[
\frac{4x^4}{x^2 + y^2} \geq 3x^2 - y^2.
\]
Taking account of similar inequalities for other pairs of variables, we obtain that
\[
\frac{4x^4}{x^2 + y^2} + \frac{4y^4}{y^2 + z^2} + \frac{4z^4}{z^2 + x^2} \geq 2(x^2 + y^2 + z^2) \geq 2(xy + yz + zx) = 2,
\]
from which we conclude that the minimum value is \(\frac{1}{2}\). This is attained when \(x = y = z = 1/\sqrt{3}\).

Solution 7. [O. Xia] Recall that, for \(r > 0, r + (1/r) \geq 2\), so that \(r \geq 2 - (1/r)\). It follows that
\[
\frac{x^4}{x^2 + y^2} = \frac{x^2}{2} \frac{2x^2}{x^2 + y^2} \geq \frac{x^2}{2} \left(2 - \frac{x^2 + y^2}{2x^2}\right) = x^2 - \frac{x^2 + y^2}{4}.
\]
Determine polynomials

(a) Let \( n \) as desired. Since \( \gcd(\, d, m \, x \) with similar equalities for the other two terms in the problem statement. Equality occurs if and only if \( x^2 = y^2 = z^2 \).

Adding the three equalities yields that Determine the minimum value of

\[
\frac{x^4}{x^2 + y^2} + \frac{y^4}{y^2 + z^2} + \frac{z^4}{z^2 + x^2} \geq \frac{x^2 + y^2 + z^2}{2}.
\]

As before, we see that the right member assumes its minimum value of \( \frac{1}{2} \) when \( x = y = z = 1/\sqrt{3} \).

651. Determine polynomials \( a(t), b(t), c(t) \) with integer coefficients such that the equation \( y^2 + 2y = x^3 - x^2 - x \) is satisfied by \( (x, y) = (a(t)/c(t), b(t)/c(t)) \).

Solution. The equation can be rewritten \( (y+1)^2 = (x-1)^2(x+1) \). Let \( x + 1 = t^2 \) so that \( y + 1 = (t^2 - 2)t \). Thus, we obtain the solution

\( (x, y) = (t^2 - 1, t^3 - 2t - 1) \).

With these polynomials, both sides of the equation are equal to \( t^6 - 4t^4 + 4t^2 - 1 \).

652. (a) Let \( m \) be any positive integer greater than 2, such that \( x^2 \equiv 1 \pmod{m} \) whenever the greatest common divisor of \( x \) and \( m \) is equal to 1. An example is \( m = 12 \). Suppose that \( n \) is a positive integer for which \( n + 1 \) is a multiple of \( m \). Prove that the sum of all of the divisors of \( n \) is divisible by \( m \).

(b) Does the result in (a) hold when \( m = 2? \)

(c) Find all possible values of \( m \) that satisfy the condition in (a).

(a) Solution 1. Let \( n + 1 \) be a multiple of \( m \). Then \( \gcd(m, n) = 1 \). We observe that \( n \) cannot be a square. Suppose, if possible, that \( n = r^2 \). Then \( \gcd(r, m) = 1 \). Hence \( r^2 \equiv 1 \pmod{m} \). But \( r^2 + 1 \equiv 0 \pmod{m} \), so that \( 2 \) is a multiple of \( m \), a contradiction.

As a result, if \( d \) is a divisor of \( n \), then \( n/d \) is a distinct divisor of \( n \). Suppose \( d | n \) (read “\( d \) divides \( n \)”). Since \( m \) divides \( n + 1 \), therefore \( \gcd(m, n) = \gcd(d, m) = 1 \), so that \( d^2 = 1 + bm \) for some integer \( b \). Also \( n + 1 = cm \) for some integer \( c \). Hence

\[
d + \frac{n}{d} = \frac{d^2 + n}{d} = \frac{1 + bm + cm - 1}{d} = \frac{(b + c)m}{d}.
\]

Since \( \gcd(d, m) = 1 \) and \( d + n/d \) is an integer, \( d \) divides \( b + c \) and so \( d + n/d \equiv 0 \pmod{m} \).

Hence

\[
\sum_{d|n} d = \sum \{(d + n/d) : d | n, d < \sqrt{n}\} \equiv 0 \pmod{m}
\]
as desired.

Solution 2. Suppose that \( m > 1 \) and \( m \) divides \( n + 1 \). Then \( \gcd(m, n) = 1 \). Suppose, if possible, that \( n = r^2 \) for some \( r \). Then, since \( \gcd(m, r) = 1, r^2 \equiv 1 \pmod{m} \). Therefore \( m \) divides both \( r^2 + 1 \) and \( r^2 - 1 \), so that \( m = 2 \). But this gives a contradiction. Hence \( n \) is not a perfect square.

Suppose that \( d \) is a divisor of \( n \). Then the greatest common divisor of \( m \) and \( d \) is 1, so that \( d^2 \equiv 1 \pmod{n} \). Suppose that \( de = n \). Then \( e \neq 1d \) and the greatest common divisor of \( m \) and \( e \) is 1. Therefore, there are numbers \( u \) and \( v \) for which both \( du \) and \( ev \) are congruent to 1 modulo \( m \). Since \( n \equiv -1 \) and \( d^2 \equiv 1 \pmod{m} \), it follows that

\[
d + e \equiv d + un \equiv u(d^2 + n) \equiv u(1 - 1) = 0 \mod{m},
\]
from which it can be deduced that \( m \) divides the sum of all the divisors of \( n \).
Solution 3. Suppose that \( n + 1 \equiv 0 \pmod{m} \). As in the first solution, it can be established that \( n \) is not a perfect square. Let \( x \) be any positive divisor of \( n \) and suppose that \( xy = n \); \( x \) and \( y \) are distinct. Since \( \gcd(x, m) = 1 \), \( x^2 \equiv 1 \pmod{m} \), so that

\[
y = x2y \equiv xn \equiv -x \pmod{m}
\]

whence \( x + y \) is a multiple of \( m \). Thus, the divisors of \( n \) comes in pairs, each of which has sum divisible by \( m \), and the result follows.

Solution 4. [M. Boase] As in the second solution, if \( xy = n \), then \( x2 \equiv y2 \equiv 1 \pmod{m} \) so that

\[
0 \equiv x2 - y2 \equiv (x - y)(x + y) \pmod{m}.
\]

For any divisor \( r \) of \( m \), we have that

\[
x(x - y) \equiv x2 - xy \equiv 2 \pmod{r}
\]

from which it follows that the greatest common divisor of \( m \) and \( x - y \) is 1. Therefore, \( m \) must divide \( x + y \) and the solution can be completed as before.

(b) Solution. When \( m = 2 \), the result does not hold. The hypothesis is true. However, the conclusion fails when \( n = 9 \) since \( 9 + 1 \) is a multiple of 2, but \( 1 + 3 + 9 = 13 \) is odd.

(c) Solution 1. By inspection, we find that \( m = 1, 2, 3, 4, 6, 8, 12, 24 \) all satisfy the condition in (a).

Suppose that \( m \) is odd. Then \( \gcd(2, m) = 1 \Rightarrow 22 = 4 \equiv 1 \pmod{m} \Rightarrow m = 1, 3 \).

Suppose that \( m \) is not divisible by 3. Then \( \gcd(3, m) = 1 \Rightarrow 9 = 32 \equiv 1 \pmod{m} \Rightarrow m = 1, 2, 4, 8 \).

Hence any further values of \( m \) not listed in the above must be even multiples of 3, that is, multiples of 6.

Suppose that \( m \geq 30 \). Then, since \( 25 = 52 \not\equiv 1 \pmod{m} \), \( m \) must be a multiple of 5.

It remains to show that in fact \( m \) cannot be a multiple of 5. We observe that there are infinitely many primes congruent to 2 or 3 modulo 5. [To see this, let \( q_1, \ldots, q_s \) be the \( s \) smallest odd primes of this form and let \( Q = 5q_1 \cdots q_s + 2 \). Then \( Q \) is odd. Also, \( Q \) cannot be a product only of primes congruent to \( \pm 1 \) modulo 5, for then \( Q \) itself would be congruent to \( \pm 1 \). Hence \( Q \) has an odd prime factor congruent to \( \pm 2 \) modulo 5, which must be distinct from \( q_1, \ldots, q_s \). Hence, no matter how many primes we have of the desired form, we can always find one more.] If possible, let \( m \) be a multiple of 5 with the stated property and let \( q \) be a prime exceeding \( m \) congruent to \( \pm 2 \) modulo 5. Then \( \gcd(q, m) = 1 \Rightarrow q2 \equiv 1 \pmod{m} \Rightarrow q2 \equiv 1 \pmod{5} \Rightarrow q \not\equiv \pm 2 \pmod{5} \), yielding a contradiction. Thus, we have given a complete collection of suitable numbers \( m \).

Solution 2. [J. Rickards] Suppose that a suitable value of \( m \) is equal to a power of 2, Then \( 32 \equiv 1 \pmod{m} \) implies that \( m \) must be equal to 4 or 8. It can be checked that both these values work.

Suppose that \( m = p^aq \), where \( p \) is an odd prime and \( p \) and \( q \) are coprime. By the Chinese Remainder Theorem, there is a value of \( x \) for which \( x \equiv 1 \pmod{q} \) and \( x \equiv 2 \pmod{p^a} \) and \( 4 \equiv x2 \equiv 1 \pmod{p^a} \). Then \( x2 \equiv 1 \pmod{m} \), so that \( 4 \equiv x2 \equiv 1 \pmod{p^a} \) and thus \( p \) must equal 3. Therefore, \( m \) must be divisible by only the primes 2 and 3. Therefore \( 25 = 52 \equiv 1 \pmod{m} \), with the result that \( m \) must divide 24. Checking reveals that the only possibilities are \( m = 3, 4, 6, 8, 12, 24 \).

Solution 3. [D. Arthur] Suppose that \( m = ab \) satisfies the condition of part (a), where the greatest common divisor of \( a \) and \( b \) is 1. Let \( \gcd(x, a) = 1 \). Since \( a \) and \( b \) are coprime, there exists a number \( t \) such that \( at \equiv 1 - x \pmod{b} \), so that \( z = x + at \) and \( b \) are coprime. Hence, the greatest common divisor of \( z \) and \( ab \) equals 1, so that \( z2 \equiv 1 \pmod{ab} \), whence \( x2 \equiv z2 \equiv 1 \pmod{a} \). Thus \( a \) (and also \( b \)) satisfies the condition of part (a).

When \( m \) is odd and exceeds 3, then \( \gcd(2, m) = 1 \), but \( 22 = 4 \not\equiv 1 \pmod{m} \), so \( m \) does not satisfy the condition. When \( m = 2^k \) for \( k \geq 4 \), then \( \gcd(3, m) = 1 \), but \( 32 = 9 \not\equiv 1 \pmod{m} \). It follows from the first
paragraph that if $m$ satisfies the condition, it cannot be divisible by a power of 2 exceeding 8 nor by an odd number exceeding 3. This leaves the possibilities 1, 2, 3, 4, 6, 8, 12, 24, all of which satisfy the condition.

**653.** Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n - r) : 1 \leq r \leq n - 1\}$. Hence $f(n) = f(1) + f(n - 1)$. Determine necessary and sufficient conditions on the pair $(a, b)$ that $f(a + b) = f(a) + f(b)$.

**Solution 1.** From the first few values of $f(n)$, we conjecture that $f(2k) = 3k$ and $f(2k + 1) = 3k + 1$ for each positive integer $k$. We establish this by induction. It is easily checked for $k = 1$. Suppose that it holds up to $k = m$.

Suppose that $2m+2$ is the sum of two positive even numbers $2x$ and $2y$. Then $f(2x) + f(2y) = 3(x + y) = 3(m + 1)$. If $2m + 2$ is the sum of two positive odd numbers $2u + 1$ and $2v + 1$, then

$$f(2u + 1) + f(2v + 1) = (3u + 1) + (3v + 1) = 3(u + v) + 2 < 3(u + v + 1) = 3(m + 1).$$

Hence $f(2m+1) = 3(m + 1)$.

Suppose $2m + 3$ is the sum of $2z$ and $2w + 1$. Then $z + w = m + 1$ and

$$f(2z) + f(2w + 1) = 3z + 3w + 1 = 3(z + w) + 1 = 3(m + 1) + 1.$$

Hence $f(2m+1) + 1 = 3(m + 1) + 1$. The conjecture is established by induction.

By checking cases on the parity of $a$ and $b$, one verifies that $f(a + b) = f(a) + f(b)$ if and only if at least one of $a$ and $b$ is even. (If $a$ and $b$ are both odd, the left side is divisible by 3 while the right side is not.)

**Solution 2.** [K. Yeats] By inspection, we conjecture that $f(n + 1) = f(n) + 2$ when $n$ is odd, and $f(n + 1) = f(n) + 1$ when $n$ is even. This is true for $n = 1, 2$. Suppose it holds up to $n = 2k$. If $2k + 1 = i + j$ with $i$ even and $j$ odd, then $f(i - 1) + f(j + 1) = f(i) + f(j) + 2 = f(i) + f(j)$ and $f(i + 1) + f(j - 1) = f(i) + f(j)$ (where defined), so in particular $f(2k + 1) = f(2k) + f(1) = f(2k) + 1$. Note that this also tells us that $f(2k + 1) = f(i) + f(j)$ whenever $i + j = 2k + 1$. Now consider $2k + 2 = i + j$. If $i$ and $j$ are both even, then

$$f(i + 1) + f(j - 1) = f(i) + 1 - f(j) - 2 = f(i) + f(j) - 1$$

while if $i$ and $j$ are both odd, then

$$f(i + 1) + f(j - 1) = f(i) + 2 - f(j) - 1 = f(i) + f(j) + 1.$$ 

Thus, $f(2k+2) = f(i) + f(j)$ if and only if $i$ and $j$ are both even. In particular, $f(2k+2) = f(2k) + f(2) = f(2k+1) - 1 + 3 = f(2k) + 2$. We thus find that $f(a + b) = f(a) + f(b)$ if and only if at least one of $a$ and $b$ is even.