Solutions for March

605. Prove that the number $299 \cdots 998200 \cdots 029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n - 1$ nines between the first 2 and the 8, and $n - 1$ zeros between the last pair of twos.

Solution. Let $a - 1, a, a + 1$ be the three consecutive numbers. The sum of their squares is $3a^2 + 2$; setting this equal to the given number yields

$$a^2 = 9 \cdot 10^{2n+1} + \cdots + 9 \cdot 10^{n+3} + 9 \cdot 10^{n+2} + 4 \cdot 10^{n+1} + 9$$

$$= (10^n - 1)10^{n+2} + 4 \cdot 10^{n+1} + 9 = 10^{2n+2} - 6 \cdot 10^{n+1} + 9$$

so that $a = 10^n - 3$.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer $n$. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed $x$.

Solution. We prove that $x_n$ is nonrational as well as positive for $n \geq 2$. Note that $x_2$ is nonrational. Suppose that $n \geq 2$ and that $x_{n+1}$ were rational; then $x_n = x_{n+1} - n^2$ would also be rational; repeating this would lead to $x_2$ being rational and a contradiction.

Observe that, for any positive integer $n \geq 2$,

$$x_n = \sqrt{x_{n-1} + (n-1)^2} > n - 1.$$ 

We prove by induction that $x_n < n$. This is true for $n = 2$. If $x_{n-1} < n - 1$, then

$$x_n^2 = x_{n-1} + (n-1)^2 < (n-1)n < n^2,$$

and the desired result follows. Thus, for each $n \geq 2$, $\lfloor x_n \rfloor = n - 1$.

For $n \geq 3$,

$$\lfloor x_n^2 \rfloor = \lfloor x_{n-1} + (n-1)^2 \rfloor = (n - 2) + (n-1)^2 = n^2 - n - 1 = n(n-1) - 1.$$ 

Therefore

$$\sum_{k=1}^{n} \lfloor x_k^2 \rfloor = \lfloor x_1^2 \rfloor + \lfloor x_2^2 \rfloor + \sum_{k=3}^{n} \lfloor x_k^2 \rfloor$$

$$= 3 + \left( \sum_{k=3}^{n} k(k-1) \right) - (n-2)$$

$$= 5 - n + \frac{1}{3} \sum_{k=3}^{n} [(k+1)k(k-1) - k(k-1)(k-2)]$$

$$= 5 - n + \frac{1}{3} [(n+1)n(n-1) - 6] = 3 - n + \frac{1}{3} (n^3 - n)$$

$$= \frac{1}{3} (n^3 - 4n + 9),$$

607. Solve the equation

$$\sin x \left( 1 + \tan x \tan \frac{x}{2} \right) = 4 - \cot x.$$ 

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Solution. For the equation to be defined, $x$ cannot be a multiple of $\pi$, so that $\sin x \neq 0$. Rearranging the terms of the equation and manipulating yields that

$$4 = \cot x + \sin x \left( \frac{\cos x \cos \frac{x}{2} + \sin x \sin \frac{x}{2}}{\cos x \cos \frac{x}{2}} \right)$$

$$= \cot x + \sin x \frac{\cos(x - (x/2))}{\cos x \cos(x/2)}$$

$$= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin 2x},$$

whence $\sin 2x = \frac{1}{2}$. Therefore $x = (-1)^k \frac{\pi}{12} + \frac{\pi}{2}$, where $k$ is an integer.

608. Find all positive integers $n$ for which $n, n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

Solution. If $n = 2$, then the numbers are 2, 5 and 11 and all are prime. Otherwise, $n$ must be odd. But in this case, the other two numbers are even exceeding 2 and so nonprime. Therefore $n = 2$ is the only possibility.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

Solution. The sum of the first nine terms of an arithmetic progression is equal to $9/2$ the sum of the first and ninth terms, from which it is seen that the ninth term is 81. Let $r$ be the common ratio of the geometric progression whose first term is 1 and whose ninth term is 81. Then $r^8 = 81$, whence $r = \pm \sqrt[3]{3}$. The sum of the first twenty terms of the geometric progression is $\frac{1}{2}(3^{10} - 1)(\pm \sqrt[3]{3} + 1)$.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$

$$\log_{10}(y^3 - y^2) = \log_5 z^2$$

$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

Solution. For $x > 1$, let

$$f(x) = 5^{\log_{10}(x^3 - x^2)}. $$

The three equations are $f(x) = y^2$, $f(y) = z^2$ and $f(z) = x^2$. Since $x^3 - x^2 = x^2(x - 1)$ is increasing, $f$ is an increasing function. If, say, $x < y$, then $y < z$ and $z < x$, yielding a contradiction. Thus, we can only have that $x = y = z$ and so

$$\log_{10}(x^3 - x^2) = \log_5 x^2.$$

Let $2t = \log_5 x^2$ so that $t > 0$, $x^2 = 5^{2t}$ and so $x = 5^t$. Therefore

$$5^{3t} - 5^{2t} = 10^{2t} \Rightarrow 5^t - 1 = 4^t \Rightarrow 5^t - 4^t = 1.$$

Since $5^t - 4^t = 4^t[(5/4)^t - 1]$ is an increasing function of $t$, we see that the equation for $t$ has a unique solution, namely $t = 1$. Therefore $x = 5$.

611. The triangle $ABC$ is isosceles with $AB = AC$ and $I$ and $O$ are the respective centres of its inscribed and circumscribed circles. If $D$ is a point on $AC$ for which $ID \parallel AB$, prove that $CI \perp OD$. 

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Solution. Since $ABC$ is isosceles, the points $A, O, I$ lie on the right bisector of $BC$. Let $AO$ meet $BC$ at $P$, $DI$ meet $BC$ at $E$, $DO$ meet $BC$ at $F$ and $CI$ meet $DF$ at $Q$.

Suppose that angle $A$ is less than $60^\circ$. Then $O$ lies between $I$ and $A$, and $Q$ lies within triangle $APB$. Since $DE \parallel AB$ and $O$ is the centre of the circumcircle of $ABC$, we have that

$$\angle CDI = \angle BAC = \angle COI,$$

so that $CIOD$ is concyclic. Therefore

$$\angle CQD = 180^\circ - (\angle QOI + \angle QIO) = 180^\circ - (\angle IDC + \angle PIC)$$
$$= 180^\circ - (\angle ICP + \angle PIC) = 90^\circ.$$

Suppose that angle $A$ exceeds $60^\circ$. Then $I$ lies between $O$ and $A$, and $Q$ lies on the same side of $AP$ as $C$. Since

$$\angle IDC + \angle IOC = \angle BAC + \angle AOC = 180^\circ,$$
the quadrilateral $IOCD$ is concyclic. Therefore

$$\angle CQD = 180^\circ - (\angle DCQ + \angle QDC) = 180^\circ - (\angle QCP + \angle ODC)$$
$$= 180^\circ - (\angle QCP + \angle OIC) = 180^\circ - (\angle ICP + \angle PIC) = 90^\circ.$$

Finally, if $\angle A = 60^\circ$, then $I$ and $O$ coincide so that $DF = DE \parallel AB$ and the result is clear.