Solutions for January

591. The point $O$ is arbitrarily selected from the interior of the angle $KAM$. A line $g$ is constructed through the point $O$, intersecting the ray $AK$ at the point $B$ and the ray $AM$ at the point $C$. Prove that the value of the expression

$$\frac{1}{[AOB]} + \frac{1}{[AOC]}$$

does not depend on the choice of the line $g$. [Note: $[MNP]$ denotes the area of triangle $MNP$.]

Solution 1. Construct a line passing through the point $O$ and parallel to $AC$. Let this line intersect the line $AB$ at the point $P$. Taking note that two triangles having their bases on a line and their third vertex on a parallel line have areas in proportion to their bases, we obtain that

$$\frac{1}{[AOB]} + \frac{1}{[AOC]} = \frac{[ABC]}{[AOB][AOC]} = \frac{[ABC]}{[AOB][AOC]}$$

$$= \frac{[ABC]}{[AOB][AOC]} \cdot \frac{[APO]}{[AOC]} = \frac{[ABC]}{[AOB]} \cdot \frac{[AOC]}{[APO]} \cdot \frac{1}{[APO]} = \frac{[ABC]}{[APO]} \cdot \frac{1}{[APO]} = \frac{1}{[APO]}.$$

Since none of the points $A, P, O$ depend on the position of the line $g$, the desired result follows.

Solution 2. Let $a = |AO|$, $b = |AB|$, $c = |AC|$, $\beta = \angle BAO$, $\gamma = \angle CAO$ and $\theta = \angle AOB$. The distance from $O$ to $AB$ is $a \sin \beta$ and from $O$ to $AC$ is $a \sin \gamma$. Therefore, $[AOB] = \frac{1}{2}ba \tan \beta$ and $[AOC] = \frac{1}{2}ca \tan \gamma$.

Note that $\angle ABO = 180^\circ - (\theta + \beta)$ and $\angle ACO = \theta - \gamma$, so that, by the Law of Sines,

$$b = \frac{a \sin \theta}{\sin(\theta + \beta)} \quad \text{and} \quad c = \frac{a \sin \theta}{\sin(\theta - \gamma)}.$$

Therefore

$$\frac{1}{[AOB]} + \frac{1}{[AOC]} = \frac{2}{ba \sin \beta} + \frac{2}{ca \sin \gamma}$$

$$= \left(\frac{2}{a^2 \sin \theta \sin \beta \sin \gamma}\right)(\sin(\theta + \beta) \sin \gamma + \sin(\theta - \gamma) \sin \beta)$$

$$= \left(\frac{2}{a^2 \sin \theta \sin \beta \sin \gamma}\right)(\sin \beta \cos \sin \gamma + \cos \theta \sin \beta \sin \gamma + \sin \theta \cos \gamma \sin \beta - \cos \theta \sin \gamma \sin \beta)$$

$$= \left(\frac{2}{a^2 \sin \beta \sin \gamma}\right)(\cos \beta \sin \gamma + \cos \gamma \sin \beta) = 2a^{-2}(\cot \beta + \cot \gamma),$$

which does not depend on the variable quantities $b, c$ and $\theta$. The result follows.

592. The incircle of the triangle $ABC$ is tangent to the sides $BC, CA$ and $AB$ at the respective points $D, E$ and $F$. Points $K$ from the line $DF$ and $L$ from the line $EF$ are such that $AK \parallel BL \parallel DE$. Prove that:

(a) the points $A, E, F$ and $K$ are concyclic, and the points $B, D, F$ and $L$ are concyclic;
(b) the points $C, K$ and $L$ are collinear.

Solution. (a) Since $AE$ is tangent to the circumcircle of triangle $DEF$ and since $AK \parallel BL$,

$$\angle AEF = \angle EDF = \angle AKF,$$

whence $A, E, F, K$ are concyclic. Since $BC$ is tangent to the circumcircle of triangle $DEF$ and since $DE \parallel BL$,

$$\angle BDF = \angle FED = \angle LED = 180^\circ - \angle BLE = 180^\circ - \angle BLF,$$
whence $B, D, F, L$ are concyclic.

(b) Since $DE \parallel AK$, $AKEF$ is a concyclic quadrilateral and $AB$ is tangent to circle $DEF$, we have that

$$\angle DEK = \angle EKA = \angle EFA = \angle EDK,$$

whence $KD = KE$. Since $DE \parallel BL$, $BLFD$ is a concyclic quadrilateral and $AB$ is tangent to circle $DEF$, we have that

$$\angle LDE = \angle BLD = \angle BFD = \angle LED,$$

whence $LD = LE$. Since $CD$ and $CE$ are tangents to circle $DEF$, $CD = CE$. Therefore, all three points $C, K, L$ lie on the right bisector of $DE$ and so are collinear.

593. Consider all natural numbers $M$ with the following properties:

(i) the four rightmost digits of $M$ are 2008;

(ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

Determine all numbers $n$ for which such numbers $M$ exist.

Solution. Since, modulo 10, squares are congruent to one of 0, 1, 4, 6, 9, and $p^n$ is square for even values of $n$, there are no even values of $n$ for which such a number $M$ exists.

Since $p^n \equiv 2008 \pmod{10^4}$ implies that $p^n \equiv 8 \pmod{16}$, we see that $p$ must be even. When $p$ is divisible by 4, then $p^n \equiv 0 \pmod{16}$ for $n \geq 2$, and when $p$ is twice an odd number, $p^n \equiv 0 \pmod{16}$ for $n \geq 4$. Therefore the only possibility for $M$ is that it be the cube of a number congruent to 2 (mod 4).

The condition that $p^3 \equiv 2008 \pmod{10^4}$ implies that $p^3 \equiv 8 \pmod{125}$. Since

$$p^3 - 8 = (p - 2)(p^2 + 2p + 4) = (p - 2)[(p + 1)^2 + 3],$$

and since the second factor is never divisible by 5 (the squares, modulo 5, are 0, 1, 4), we must have that $p \equiv 2 \pmod{125}$. Putting this together with $p$ being twice an odd number, we find that the smallest possibilities are equal to 502 and 1002.

We have that $502^3 = 126506008$ and $1002^3 = 1006012008$. Thus, such numbers $M$ exist if and only $n = 3$.

594. For each natural number $N$, denote by $S(N)$ the sum of the digits of $N$. Are there natural numbers $N$ which satisfy the condition severally:

(a) $S(N) + S(N^2) = 2008$;

(b) $S(N) + S(N^2) = 2009$?

Solution. We have that

$$S(N) + S(N^2) \equiv N + N^2 = (N + 1) \pmod{9}$$

(mod 9). This number is congruent to either 0 or 2, modulo 3. In particular, it can never assume the value of 2008, which is congruent to 1, modulo 3.

For part (b), we try a number $N$ of the form

$$N = 1 + 10^3 + 10^6 + \cdots + 10^{3r},$$

where $100 \leq r \leq 999$. Then $S(N) = r + 1$,

$$N^2 = 1 + 2 \cdot 10^3 + 3 \cdot 10^6 + \cdots + r \cdot 10^{3r-1} + (r + 1) \cdot 10^r + r \cdot 10^{r+1} + \cdots + 2 \cdot 10^{6r-1} + 10^{6r}$$

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and, since each coefficient of a power of 10 has at most three digits and there is no carry to a digit arising from another power,

\[ S(n^2) = 2 \sum_{k=1}^{r} S(k) + S(r + 1) = 2 \sum_{k=1}^{99} S(k) + 2 \sum_{k=101}^{r} S(k) + S(r + 1). \]

The numbers less than 100 have 200 digits in all (counting 0 as the first digit of single-digit numbers), each appearing equally often (20 times), so that

\[ 2 \sum_{k=1}^{99} S(k) = 2[20(1 + 2 + \cdots + 9)] = 1800. \]

Now let \( r = 108 \). Then \( S(100) + S(101) + S(108) = 9 + 36 = 45 \), so that, when \( N = 1001001 \cdots 1001 \) with 109 ones interspersed by double zeros,

\[ S(N) + S(N^2) = 109 + 1800 + 90 + 10 = 2009. \]

Therefore, the equation in (b) is solvable for some natural number \( N \).

595. What are the dimensions of the greatest \( n \times n \) square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (i.e. that share a side) differ by 1?

Solution. We begin by establishing some restrictions. The parity of the number of coins in any two adjacent cells differ, so that at least one of any pair of adjacent cells contains at least one coin. This ensures that the number of cells cannot exceed \( 2 \times 111 + 1 = 2\times 15^2 \), so that \( n \leq 14 \). Since there are 111 cells, there must be an odd number of cells that contain an odd number of coins. Since in a \( 14 \times 14 \) chessboard, there must be \( 98 = \frac{1}{2} \times 196 \) cells with an odd number of coins, \( n = 14 \) is not possible.

We show that a \( 13 \times 13 \) chessboard admits a suitable placement of coins. Begin by placing a single coin in every second cell so that each corner cell contains one coin. This uses up 85 coins. Now place two coins in each of thirteen of the remaining 84 vacant cells. We have placed \( 85 + 26 = 111 \) coins in such a ways as to satisfy the condition.

Hence, a \( 13 \times 13 \) chessboard is the largest that admits the desired placement.

596. A \( 12 \times 12 \) square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size \( 1 \times 3 \) without overlapping any of them?

Solution. Let the major diagonal in question go from upper left to lower right. Label the cells by letters \( A, B, C \) with \( A \) in the upper left corner, so that \( ABC \) appears in this cyclic order across each row and \( ACB \) appears in this cyclic order down each column. There are thus 48 occurrences of each label, and each cell of the major diagonal is labelled with an \( A \). Since each horizontal or vertical placement of \( 1 \times 3 \) tiles must cover one cell with each label, any placement of any number of such tiles must cover equally many cells of each label. However, removing three cells down the major diagonal removes three cells of a single label and leaves of dearth of cells with label \( A \). Therefore, a covering of the remaining 141 cells with 47 tiles is not possible.

597. Find all pairs of natural numbers \( (x, y) \) that satisfy the equation

\[ 2x(xy - 2y - 3) = (x + y)(3x + y). \]
Solution. The given equation can be rewritten as a quadratic in $y$:

$$y^2 + (8x - 2x^2)y + (3x^2 + 6x) = 0.$$  

Its discriminant is equal to

$$(64x^2 - 32x^3 + 4x^4) - 4(3x^2 + 6x) = 4x(x^3 - 8x^2 + 13x - 6) = 4x(x - 6)(x - 1)^2.$$  

For there to be a solution in integers, it is necessary that this discriminant be a perfect square. This happens if and only of

$$z^2 = x(x - 6) = (x - 3)^2 - 9,$$

or

$$9 = (x - 3)^2 - z^2 = (x + z - 3)(x - z - 3),$$

for some integer $z$. Checking all the factorizations $9 = (-9) \times (-1) = (-3) \times (-3) = (-1) \times (-9) = 9 \times 1 = 3 \times 3 = 1 \times 9$, we find that $(x, z) = (-2, \pm 4), (0, 0), (8, \pm 4), (6, 0)$.

This leads to a complete solutions set in integers:

$$(x, y) = (-2, 0), (-2, -8), (-0, 0), (8, 4), (8, 60), (6, 12).$$

Therefore, the only solutions in natural numbers to the equation are

$$(x, y) = (6, 12), (8, 4), (8, 60),$$

all of which check out.