Solutions for September

563. (a) Determine infinitely many triples \((a, b, c)\) of integers for which \(a, b, c\) are not in arithmetic progression and \(ab + 1, bc + 1, ca + 1\) are all squares.

(b) Determine infinitely many triples \((a, b, c)\) of integers for which \(a, b, c\) are in arithmetic progression and \(ab + 1, bc + 1, ca + 1\) are all squares.

(c) Determine infinitely many triples \((u, v, w)\) of integers for which \(uw - 1, vw - 1, wu - 1\) are all squares. (Can it be arranged that \(u, v, w\) are in arithmetic progression?)

Solution. (a) Here are some families of solutions that are (mostly) not in arithmetic progression, where \(n\) is an integer:

\[(0, 0, n); (0, n - 1, n + 1); (0, 2, 2n(n + 1)); (1, n^2 - 1, n^2 + 2n); (n - 1, n + 1, 4n); (n, n + 2, 4(n + 1)); (m, mn^2 + 2n, m(n + 1)^2 + 2(n + 1)); (f_2(n - 1), f_2n, f_2(n + 1)) .\]

Here, \(\{f_n\}\) is the Fibonacci sequence defined by \(f_1 = f_2 = 1\) and \(f_{n+1} = f_n + f_{n-1}\) for every integer \(n\). We need to establish that \(f_{2n}f_{2n+2} + 1 = f_{2n+1}^2\) and \(f_{2n-2}f_{2n+2} + 1 = f_{2n}^2\) for each integer \(n\). Since

\[f_{2n+1}^2 - f_{2n}^2 = f_{2n+2}(f_{2n} - f_{2n-2}) = (f_{2n}f_{2n+2} + 1) - (f_{2n-2}f_{2n+2} + 1),\]

the two equations are equivalent. Note that

\[f_{2n}f_{2n+2} - f_{2n+1}^2 = f_{2n}(f_{2n+1} + f_{2n}) - f_{2n+1}^2 = f_{2n+1}(f_{2n} - f_{2n+1}) + f_{2n}^2 = -(f_{2n+1}f_{2n-1} - f_n^2) ;\]

a proof by induction can be devised for the first equation.

(b) (i) Some examples for \((a, b, c)\) are \((-1, 0, 1), (0, 2, 4), (1, 8, 15), (4, 30, 56), (15, 112, 209)\). This suggests the possibility \((u_n, 2u_{n+1}, u_{n+2})\) where \(u_0 = 0, u_1 = 1, u_2 = 4\) and \(u_{n+1} = 4u_{n-1} - u_n\) for integral \(n\). Since \(u_{n+1} - 2u_n = 2u_n - u_{n-1}, u_{n-1}, u_n, u_{n+1}\) are in arithmetic progression.

We now prove, for each integer \(n\),

\[2u_nu_{n+1} + 1 = (u_{n+1} - u_n)^2 \quad (1)\]

\[u_{n+2}u_n + 1 = u_{n+1}^2 \quad (2)\]

\[2u_{n+1}u_{n+2} + 1 = (u_{n+2} - u_{n+1})^2 \quad (3)\]

Properties (1) and (3) are the same. The truth of (2) is equivalent to the truth of (1), since

\[[(2u_nu_{n+1} + 1) - (u_{n+1} - u_n)^2] + [(u_nu_{n+2} + 1) - u_{n+1}^2]\]

\[= u_n(2u_{n+1} - u_{n+2}) + u_n(2u_{n+1} - u_n)\]

\[= -u_n(u_{n+2} - 4u_{n+1} + u_n) = 0 .\]

We establish (2) by induction. Since

\[u_{n+2}u_n + 1 - u_{n+1}^2 = u_n(4u_{n+1} - u_n) + 1 - u_{n+1}^2\]

\[= u_{n+1}(4u_n - u_{n+1}) + 1 - u_n^2\]

\[= u_{n+1}u_{n-1} + 1 - u_n^2 ,\]

\[u_{n+2}u_n + 1 - u_{n+1}^2 = u_2u_0 + 1 - u_1^2 = 0\] for all \(n\). The desired results follow.
(b) (ii) [A. Dhawan] Let \( v^2 - 3u^2 = 1 \) for some integers \( v \) and \( u \). Then, if \((a, b, c) = (2u - v, 2u, 2u + v)\), then

\[
\begin{align*}
ab + 1 &= (2u - v)2u + 1 = 4u^2 - 2uv + 1 \\
&= u^2 + (v^2 - 1) - 2uv + 1 = (u - v)^2; \\
bv + 1 &= 2u(2u + v) + 1 = 4u^2 + 2uv + 1 \\
&= u^2 + 2uv + v^2 - 1 + 1 = (u + v)^2;
\end{align*}
\]

and

\[
ac + 1 = (2u - v)(2u + v) + 1
\]

\[
= 4u^2 - v^2 + 1 = u^2. 
\]

(Note that in this solution, the roots of the square, not all positive, are also in arithmetic progression.)

The equation \( v^2 - 3u^2 = 1 \) is a Pell’s equation with infinitely many solutions given by \((v, u) = (x_n, y_n)\), where \( x_n + y_n\sqrt{3} = (2 + \sqrt{3})^n \), for positive integers \( n \).

(b) (iii) We look for solutions in which one integer is 0. Thus \((a, b, c)\) has the form \((0, p, 2p)\), where \(2p^2 + 1 = q^2\). This is a Pell’s equation whose solutions are given by \((q, p) = (x_n, y_n)\) where \( x_n + y_n\sqrt{2} = (3 + \sqrt{2})^n \) for positive integers \( n \). P. Wen also identified triples \((0, p, 2p)\) where \(2p^2 + 1\) is square. Since this square is odd, it must have the form \((2y + 1)^2\), so that \(p^2 = 2y(y + 1)\). Thus, \(p\) is even, say \(p = 2x\), and so \(x^2 = \frac{1}{2}y(y + 1)\), which is at once a square and a triangular number. Conversely, any triangular number which is also a square gives a solution triple, so we need to know that there are infinitely many such. If \(x^2 = \frac{1}{2}y(y + 1)\), then \(8x^2 + 1 = (2y + 1)^2\), so that

\[
[x(2y + 1)]^2 = 4p^2(2y + 1)^2 = \frac{4y(y + 1)(2y + 1)^2}{2}
\]

\[
= \frac{1}{2}(4y^2 + 4y)(4y^2 + 4y + 1).
\]

Starting with \((x, y) = (1, 1)\), we are led to \((6, 8), (204, 288)\) and so on, and obtain the solutions \((a, b, c) = (0, 2, 4), (0, 12, 24), (0, 408, 816), \cdots\).

(c) Here are some families of solutions for \((u, v, w)\): \((1, 1, n^2 + 1), (1, n^2 + 1, (n + 1)^2 + 1)\) along with \((f_{2n-1}, f_{2n+1}, f_{2n+3})\), where \(f_n\) is the Fibonacci sequence defined in the solution to (a). S.H. Lee found a two-parameter family:

\[
(m^2 + 1, (m^2 + 1)n^2 + 2mn + 1, (m^2 + 1)(n + 1)^2 + 2m(n + 1) + 1). 
\]

In this case,

\[
\begin{align*}
uv - 1 &= [(m^2 + 1)n + m^2]^2; \\
vw - 1 &= [(m^2 + 1)(n + 1) + m]^2; \\
uw - 1 &= [(m^2 + 1)n(n + 1) + 2mn + m + 1]^2.
\end{align*}
\]

J. Zung identified the triple \((1, 2, p^2 + 1)\) where \(q^2 - 2p^2 = 1\) for some integer \(q\). [Exercise: Check these out.]

564. Let \(x_1 = 2\) and

\[
x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}
\]

for \(n \geq 1\). Prove that, for all \(n > 1\), \(1 < x_n < 2\).

Solution 1. Since, for \(n \geq 1\),

\[
x_{n+1} - 1 = \frac{2x_n}{3} + \frac{1}{3x_n} - 1 = \frac{2x_n^2 - 3x_n + 1}{3x_n} = \frac{(2x_n - 1)(x_n - 1)}{3x_n},
\]

it can be shown by induction that \(x_n > 1\) for all \(n \geq 1\).
Since, for \( n \geq 1 \),
\[
x_n - x_{n+1} = \frac{x_n^2 - 1}{3x_n},
\]
it follows that \( x_{n+1} < x_n \leq 2 \) for all \( n \geq 1 \). Thus we have the desired inequality.

**Solution 2.** Observe that \( x_2 = 3/2 \). As an induction hypothesis, suppose that \( 1 < x_n < 2 \) for some \( n \geq 2 \). Then
\[
x_{n+1} = \frac{x_n}{3} + \frac{1}{3} \left( x_n + \frac{1}{x_n} \right) > \frac{1}{3} + \frac{1}{3} \cdot 2 = 1
\]
by the arithmetic-geometric means inequality. Also,
\[
x_{n+1} = \frac{2x_n}{3} + \frac{1}{3} \left( x_n + \frac{1}{x_n} \right) > \frac{4}{3} + \frac{1}{3} = \frac{5}{3} < 2.
\]
The result follows.

**Comment.** An induction argument for the right inequality can be based on the observation that, when \( 1 < x < 2 \), the quadratic \( 2x^2 - 6x + 1 = 2(x - \frac{3}{2})^2 - \frac{7}{2} \leq \frac{7}{2} - \frac{7}{2} = -3 < 0 \), whence \( 2x^2 + 1 < 6x \) and \( (2x_n/3 + (1/3x_n))^2 < 2 \).

**Solution 3.** Let \( f(x) = (2x/3) + (1/3x) \). Then \( f'(x) = (2/3) - (1/3x^3) \geq 1/3 > 0 \) when \( 1 < x \). Hence \( f(x) \) is strictly increasing on the interval \([1, 2]\) and so takes values strictly between \( f(1) = 1 \) and \( f(2) = 3/2 \) on the open interval \((0, 1)\). Since \( x_2 \in (0, 1) \) and \( x_{n+1} = f(x_n) \) for \( n \geq 1 \), the desired result can be established by induction.

565. Let \( ABC \) be an acute-angled triangle. Points \( A_1 \) and \( A_2 \) are located on side \( BC \) so that the four points are ordered \( B, A_1, A_2, C \); similarly \( B_1 \) and \( B_2 \) are on \( CA \) in the order \( C, B_1, B_2, A \) and \( C_1 \) and \( C_2 \) on side \( AB \) in order \( A, C_1, C_2, B \). All the angles \( AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 \) are equal to \( \theta \). Let \( \Sigma_1 \) be the triangle bounded by the lines \( AA_1, BB_1, CC_1 \) and \( \Sigma_2 \) the triangle bounded by the lines \( AA_2, BB_2, CC_2 \). Prove that all six vertices of the triangles are concyclic.

**Solution 1.** Let \( A_0B_0C_0 \) be the triangle with \( B_0C_0\|BC, C_0A_0\|CA, A_0B_0\|AB \) where \( A, B, C \) are the respective midpoints of \( B_0C_0, C_0A_0, A_0B_0 \). Then the orthocentre \( H \) of triangle \( ABC \) is the circumcentre of triangle \( A_0B_0C_0 \).

Suppose that \( K \) is the intersection point of \( AA_2 \) and \( BB_2 \). Since the exterior angle at \( A_2 \) is equal to the interior angle at \( B_2 \), the quadrilateral \( A_2KB_2C \) is concyclic, so that \( \angle BKA_2 = \angle BCA = \angle BC_0A \). Therefore, the quadrilateral \( A_0KBK \) is concyclic; the quadrilateral \( A_0BH \) with right angles at \( A \) and \( B \) is concyclic. Thus, \( BC_0AKH \) is concyclic and so \( \angle C_0KH = \angle C_0AH = 90^\circ \).

Since \( C_0A_0\|AC, \angle C_0HK = \angle C_0BK = \angle BB_2C = \theta \). Therefore \( |HK| = |HC_0| \cos \theta = R \cos \theta \), where \( R \) is the circumradius of triangle \( A_0B_0C_0 \). The same argument can be applied to the intersection point of any pairs \( (AA_i, BB_i), (BB_i, CC_i), (CC_i, AA_i) \) \( (i = 1, 2) \). All the vertices lie on the circle with centre \( H \) and radius \( R \).

**Solution 2.** [A. Murali] Let \( AA_1 \cap BB_1 = P, BB_1 \cap CC_1 = Q, CC_1 \cap AA_1 = R, AA_2 \cap BB_2 = V, BB_2 \cap CC_2 = W, CC_2 \cap AA_2 = U \). We have that
\[
\angle A_2CU = \angle BCC_2 = 180^\circ - \angle ABC - \angle BC_2C = 180^\circ - \angle ABC - (180^\circ - \theta) = \theta - \angle ABC,
\]
and
\[
\angle CUA_2 = 180^\circ - (\angle A_2CU + \angle AA_2C) = 180^\circ - (\theta - \angle ABC) - (180^\circ - \theta) = \angle ABC.
\]
Since $\angle AA_1B = \angle CA_2U$ and $\angle ABA_1 = \angle ABC = \angle A_2UC$, triangles $AA_1B$ and $CA_2U$ are similar. Therefore $CA_2 : A_2U = AA_1 : BA_1$, from which

$$|A_2U| = \frac{|CA_2| \times |BA_1|}{|AA_1|}.$$  

Similarly, $\angle BPA_1 = \angle BCA$, which along with $\angle BA_1A = \angle BB_1C$ implies that triangles $BA_1P$ and $BB_1C$ are similar. Therefore

$$|PA_1| = \frac{|BA_1| \times |B_1C|}{|BB_1|}.$$  

Hence,

$$\frac{|A_2U|}{|A_1P|} = \frac{|CA_2| \times |BB_1|}{|AA_1| \times |B_1C|} = \frac{|CA_2| \times |BB_1|}{|AA_2| \times |B_1C|}.$$  

Since triangles $CBB_1$ and $CAA_2$ are similar, $CA_2 : AA_2 = CB_1 : BB_1$, from which it follows that $UA_2 = PA_1$, so that $UA_2 : AA_2 = PA_1 : AA_1$ and $PU \parallel |A_1A_2|.$

Similarly, $QV \parallel B_1B_2$. Therefore

$$\angle PUV = \angle A_1A_2A = \theta = \angle B_2B_1B = \angle VQP$$  

and $PUQV$ is concyclic.

Since $\angle C_2UA = \angle CUA_2 = \angle ABC$ and $\angle AC_2U = \theta = \angle BC_1C$, triangles $AUC_2$ and $CBC_1$ are similar, so that

$$|UC_2| = \frac{|BC_1| \times |C_2A|}{|C_1C|}.$$  

Since triangles $BQC_1$ and $CAC_2$ are similar,

$$|QC_1| = \frac{|BC_1| \times |AC_2|}{|CC_2|}.$$  

Since $CC_2 = CC_1$, $UC_2 = QC_1$ so that $UQ \parallel BA$. Similarly $WP \parallel AC$. Therefore, $\angle WPQ = 180^\circ - \theta = \angle WUQ$, and $WP\parallel$ is concyclic.

Since $RWPU$, $WPUQ$ and $PUQV$ are all concyclic, $R$ and $Q$ lie on the circle through $W, P, U$ and $W$ and $V$ lie on the circle through $P, U, Q$. The result follows.

Solution 3. [P. Wen] Use the notation of Solution 2 and let $H$ denote the orthocentre of triangle $ABC$. Since $\angle WBH = 90^\circ - \theta = \angle WCH$, the points $B, W, H, C$ are concyclic; similarly, $B, H, Q, C$ are concyclic. Hence $B, W, H, Q, C$ are concyclic. Similarly, $A, V, H, P, B$ are concyclic.

Since

$$\angle PQW = \angle BQW = \angle BCW = \angle BCC_2 = \angle BAA_1 = \angle BAP = \angle BVP = \angle PVW,$$

the points $P, Q, V, W$ are concyclic. Since $\angle BHW = \angle BCW = \angle BAP = \angle BHP$, $\angle HBW = \angle HBB_2 = \angle HBB_1 = \angle HBP$, and side $BH$ is common, the triangles $BHW$ and $BHP$ are congruent, so that $BP = BW$.

Since $\angle PHW = 2\angle BHW = 2\angle PQW$, $H$ must be the centre of the circle through $P, Q, V, W$, so that $H$ is equidistant from these four points. Similarly, $H$ is equidistant from the four points $R, P, U, V$ and from the points $Q, R, W, U$. The desired result follows.

Solution 4. [P.J. Zhao] Use the notation of Solutions 2 and 3, with $H$ the orthocentre of triangle $ABC$. Since the quadrilaterals $BC_1RA_1$, $BC_1B_2C$ and $CA_2VB_2$ are concyclic, we have that

$$AR : AA_1 = AC_1: AB = AB_2 : AC = AV : AA_2.$$
Since $AA_1 = AA_2$, $AR = AV$. As $AA_1A_2$ is isosceles, $AH$ bisects angle $A_1AA_2$ and triangles $AHR$ and $AHV$ are congruent (SAS), so that $HR = HV$. Similarly, $HP = HW$ and $HQ = HU$.

Since the quadrilaterals $CB_1PA_1$, $BC_2B_1C$ and $BC_2UA_2$ are concyclic, it follows that

$$AP : AA_1 = AB_1 : AC = AC_2 : AB = AU : AA_2,$$

whence $AP = AU$. Since triangles $AHP$ and $AHU$ are congruent, $HP = HU$. Similarly, $HQ = HV$ and $HR = HW$.

Thus, all six vertices of the two triangles are equidistant from $H$ and the result follows.

Comment. J. Zung observed that a rotation about $H$ through the angle $2\theta$ takes the line $AA_1$ onto the line $A_2A$, the line $BB_1$ onto the line $B_2B$ and the line $CC_1$ onto the line $C_2C$. To see this, note that if $A_3$ and $A_4$ are the feet of the perpendiculars dropped from $H$ to $AA_1$ and $AA_2$ respectively, then

$$\angle A_3HA_4 = \angle A_3HA + \angle AHA_4 = \angle AA_1A_2 + \angle AA_2A_1 = 2\theta.$$ 

This rotation takes $P \to V$, $Q \to W$, $R \to U$, so that $HP = HV$, $HQ = HW$, $HR = HU$. This taken with either half of the argument of Solution 4 yields the result.

566. A deck of cards numbered 1 to $n$ (one card for each number) is arranged in random order and placed on the table. If the card numbered $k$ is on top, remove the $k$th card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

Solution For each card, a move must result in exactly one of the following possibilities: (i) the card remains in the same position; (ii) the card moves one position lower in the deck; (iii) the card is brought to the top of the deck.

We prove by induction the following statement: Suppose that we have deck of $m$ cards each with a different number, and that we follow the procedure of the problem; then after at most $2^{m-1} - 1$ moves the process will have to stop either because card 1 comes to the top or a card with a number exceeding $m$ comes to the top. It is straightforward to see that the result holds for $m = 1$ and $m = 2$. Suppose that when $1 \leq m \leq r - 1$.

Let $m = r$. Since there are $r$ cards with different numbers, there is a card $u$ where either $u = 1$ or $u > r$. Suppose that $u$ occurs in the $k$th position. Then the first $k - 1$ positions must contain card 1 or a card exceeding $k - 1$. By the induction hypothesis, in at most $2^{k-2} - 1$ moves one of the following must occur: (1) the process stops because a card numbered 1 or with a number exceeding $m$ (possibly $u$) comes to the top, or (2) a card with a number between $k + 1$ and $m$ inclusive comes to the top. In the second case, one more move will cause $u$ to go to the $(k+1)$th position. Therefore, after at most $1 + 2 + \cdots + 2^{r-3} = 2^{r-2} - 1$, either the process has stopped or $u$ has been forced from the $(r-1)$th position to the $r$th position.

The top $r - 1$ cards must contain at least one lying outside of the range $[2, r - 1]$. Therefore, in at most $2^{r-2} - 1$ further moves, either the process stops, because card number 1 or a card with a number exceeding $r$ comes to the top, or else $r$ comes to the top. In the latter case, one further move will make $u$ come to the top. Thus, we can get a card with either the number 1 or a card exceeding $m$ to the top in at most $(2^{r-2} - 1) + (2^{r-2} - 1) + 1 = 2^{r-1} - 1$ moves.

The desired result is a special case of this, where $m = n$ and the card outside of the range $[2, n]$ is the card numbered 1.

There is an initial arrangement of the cards where the maximum number of moves is attained, namely $(n, 1, n - 1, n - 2, \cdots, 3, 2)$. To show this, we establish the following result:

Let $m \geq 2$. Then the sequence $(m, u, m - 1, m - 2, \cdots, 2)$ becomes the sequence $(u, m, m - 1, m - 2, \cdots, 2)$ in exactly $2^{m-1} - 1$ moves, where $u$ is any number.
This is true for \( m = 2 \) \((2, u) \rightarrow (u, 2)\) and \( m = 3 \) \((3, u, 2) \rightarrow (2, 3, u) \rightarrow (3, 2, u) \rightarrow (u, 3, 2)\). Assume that \( m \geq 4 \) and that the result holds for all values of \( m \) up to and including \( k - 1 \). Then we can use the induction hypothesis to make changes as follows (where the number in square brackets indicates the number of moves):

\[
(k, u, k - 1, \ldots, 2) \rightarrow [1] \rightarrow (2, k, u, k - 1, \ldots, 3) \rightarrow [1] \rightarrow (k, 2, u, k - 1, \ldots, 3)
\]
\[
\rightarrow [1] \rightarrow (3, k, 2, u, k - 1, \ldots, 4) \rightarrow [3] \rightarrow (k, 3, 2, u, k - 1, \ldots, 4)
\]
\[
\rightarrow [1] \rightarrow (4, k, 3, 2, u, k - 1, \ldots, 5) \rightarrow [7] \rightarrow (k, 4, 3, 2, u, k - 1, \ldots, 5)
\]
\[
\vdots
\]
\[
\rightarrow [1] \rightarrow (j, k, j - 1, \ldots, 2, u, k - 1, \ldots, j + 1)
\]
\[
\rightarrow [2^{j-1} - 1] \rightarrow (k, j, j - 1, \ldots, 2, u, k - 1, \ldots, j + 1)
\]
\[
\vdots
\]
\[
\rightarrow [1] \rightarrow (k - 1, k, k - 2, \ldots, 3, 2, u) \rightarrow [2^{k-2} - 1] \rightarrow (k - 1, k - 2, \ldots, 3, 2, u)
\]
\[
\rightarrow [1] \rightarrow (u, k, k - 1, \ldots, 3, 2)
\]

The total number of moves is

\[
1 + \sum_{j=2}^{k-2} [(2^{j-1} - 1) + 1] = 1 + 2 + \cdots + 2^{k-2} = 2^{k-1} - 1.
\]

In particular, when \( u = 1 \) and \( k = n \), we conclude that \((n, 1, n - 1, \ldots, 2)\) goes to \((1, n, n - 1, \ldots, 2)\) in \(2^{n-1} - 1\) moves.

Comment. A. Abdi provided the following induction argument that the process must terminate. The result clearly holds for \( n = 1 \). Suppose it holds for \( 1 \leq n \leq m - 1 \). If card 1 never comes to the top, then the process never terminates and card 1 eventually finds its way to position \( r \leq m \) and stays there. The cards below position \( r \) (if any) never move from that point on. Let \( X \) be the set of cards on top of 1 at that point whose numbers exceed \( r \) and \( Y \) the set of cards on top of 1 whose numbers do not exceed \( r \), so that \( \#X + \#Y = r - 1 \). Since card 1 cannot move down, the cards in \( X \) never come to the top, so it is immaterial what numbers appear on these cards. Relabel these cards with numbers from \{2, 3, 4, \ldots, r\} that do not belong to the cards in \( Y \), so that the numbers from 2 to \( r \) inclusive all appear on top of card 1. These cards get permuted among themselves by subsequent moves.

However, by the induction hypothesis applied to this deck of \( r - 1 \leq m - 1 \) cards atop card 1 (with card \( r \) relabelled to a second card 1), we see that card \( r \) must eventually come to the top, when then will force card 1 to come to the top. This yields a contradiction of the assertion that the process can go on forever.

567. (a) Let \( A, B, C, D \) be four distinct points in a straight line. For any points \( X, Y \) on the line, let \( XY \) denote the directed distance between them. In other words, a positive direction is selected on the line and \( XY = \pm |XY| \) according as the direction \( X \) to \( Y \) is positive or negative. Define

\[
(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA}.
\]

Prove that \((AB, CD) + (AC, BD) = 1\).

(b) In the situation of (a), suppose in addition that \((AC, BD) = -1\). Prove that

\[
\frac{1}{AC} = \frac{1}{2} \left( \frac{1}{AB} + \frac{1}{AD} \right),
\]
and that
\[ OC^2 = OB \times OD, \]
where \( O \) is the midpoint of \( AC \). Deduce from the latter that, if \( Q \) is the midpoint of \( BD \) and if the circles on diameters \( AC \) and \( BD \) intersect at \( P, \angle OPQ = 90^\circ \).

(c) Suppose that \( A, B, C, D \) are four distinct on one line and that \( P, Q, R, S \) are four distinct points on a second line. Suppose that \( AP, BQ, CR \) and \( DS \) all intersect in a common point \( V \). Prove that \( (AC, BD) = (PR, QS) \).

(d) Suppose that \( ABQP \) is a quadrilateral in the plane with no two sides parallel. Let \( AQ \) and \( BP \) intersect in \( U \), and let \( AP \) and \( BQ \) intersect in \( V \). Suppose that \( VU \) and \( PQ \) produced meet \( AB \) at \( C \) and \( D \) respectively, and that \( VU \) meets \( PQ \) at \( W \). Prove that \( (AB, CD) = (PQ, WD) = -1 \).

Solution. (a)
\[
\frac{AC \times BD}{CB \times DA} + \frac{AB \times CD}{BC \times DA} = \frac{(AB + BC) \times (BC + CD) - AB \times CD}{BC \times AD}
\]
\[
= \frac{BC \times (AB + BC + CD)}{BC \times AD} = 1.
\]

(b) \( AB \times CD = BC \times AD \implies AB \times (AD + CA) = (BA + AC) \times AD \implies 2AB \times AD = AB \times AC + AC \times AD \)
\[
\implies \frac{1}{AC} = \frac{1}{2} \left( \frac{1}{AB} + \frac{1}{AD} \right).
\]
Since \( AB = AO + OB = OC + OB, AD = AO + OD = OC + OD \) and \( AC = 2OC \),
\[
\frac{1}{OC} = \frac{1}{OB + OC} + \frac{1}{OD + OC},
\]
from which the desired result follows. Since \( OP = OC^2, OP^2 = OB \times OD \), so that \( OP \) is tangent to the circle of diameter \( BD \). Hence \( PQ \perp OP \) and the result follows.

Comment. For the last part, M. Sardarli noted that
\[
OP^2 + PQ^2 = OC^2 + BQ^2 = OB \times OD + BQ^2 = (OQ + QB)(OQ - QB) + BQ^2
\]
\[ = OQ^2 - QB^2 + BQ^2 = OQ^2, \]
whence \( \angle OPQ = 90^\circ \).

(c) First observe that, of both lines lie on the same side of \( V \), then corresponding lengths among \( A, B, C, D \) and \( P, Q, R, S \) have the same signs, while if \( V \) is between the lines, then the signs are opposite. Let \( a, b, c, d \) be the respective lengths of \( AV, BV, CV, DV \); let \( \alpha, \beta, \gamma, \delta \) be the respective angles \( AVB, CVD, BVC, DVA \); let \( h \) be the distance from \( V \) to the line \( ABCD \). Then
\[
\frac{|(AC, BD)|}{(AB, CD)} = \frac{|AB \times CD|}{BC \times DA} = \frac{|(\frac{1}{2}h \times AB) \times (\frac{1}{2}h \times CD)|}{(\frac{1}{2}h \times BC) \times (\frac{1}{2}h \times DA)}
\]
\[
= \frac{|AVB| \times |CVD|}{|BTC| \times |DTA|} = \frac{(\frac{1}{2}ab \sin \alpha)(\frac{1}{2}cd \sin \beta)}{(\frac{1}{2}bc \sin \gamma)(\frac{1}{2}ad \sin \delta)}
\]
\[
= \frac{\sin \alpha \sin \beta}{\sin \gamma \sin \delta}.
\]
Since $\angle AVB = \angle PVQ$, etc., we find that $|(PR, QS)| = (\sin \alpha \sin \beta)/(\sin \gamma \sin \delta)$, and the result follows.
(d) By (c), with the role of $V$ played respectively by $V$ and $U$, we obtain that

$$(AB, CD) = (PQ, WD) = (BA, CD) = \frac{1}{(AB, CD)},$$

so that $(AB, CD)^2 = 1$. Since $(AB, CD) + (AC, BD) = 1$ and $(AC, BD)$ can vanish only if $A = B$ or $C = D$, we must have that $(AB, CD) = -1$.

568. Let $ABC$ be a triangle and the point $D$ on $BC$ be the foot of the altitude $AD$ from $A$. Suppose that $H$ lies on the segment $AD$ and that $BH$ and $CH$ intersect $AC$ and $AB$ at $E$ and $F$ respectively.

Prove that $\angle FDH = \angle HDE$.

Solution 1. Suppose that $ED \parallel AB$. Then by Ceva’s theorem,

$$1 = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CE|}{|EA|} = \frac{|AF|}{|FB|} \cdot \frac{|BD|}{|DC|} \cdot \frac{|CD|}{|DB|},$$

so that $AF = FB$. Hence $F$ is the circumcentre of the right triangle $ADB$, so that $AF = DF$ and $\angle FDB = \angle FAD = \angle HDE$.

Otherwise, let $AB$ and $ED$ produced intersect at $G$. Then, in the notation of problem 567, $(AB, FG) = -1$. Therefore

$$\sin \angle ADF \sin \angle BDG = \sin \angle FDB \sin \angle ADG = \cos \angle ADF \cos \angle BDG.$$ 

Hence $\tan \angle ADF = \cot \angle BDG = \tan \angle ADE$ and $\angle ADF = \angle ADE$.

Solution 2. Suppose that $DE$ and $DF$ intersect the line through $A$ parallel to $BC$ at the points $M$ and $N$ respectively. Since triangles $BDF$ and $ANF$ are similar, as are triangles $CDE$ and $AME$,

$$\frac{|AM|}{|AN|} = \frac{|AM|}{|CD|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|DB|}{|AN|} = \frac{|AE|}{|EC|} \cdot \frac{|CD|}{|DB|} \cdot \frac{|BF|}{|AF|} = 1,$$

by Ceva’s theorem. Therefore, $AM = AN$, so that triangles $AMD$ and $AND$ are congruent and $\angle ADF = \angle ADM$.

Solution 3. [R. Peng] Suppose that the points $K$ and $L$ are selected on $BC$ so that $EK \perp CB$ and $FL \perp BC$. Let $X = CH \capEK$ and $Y = BH \cap FL$. Then $FL\|EK$, so that the triangles $FYH$ and $XEH$ with respective heights $LD$ and $KD$ are similar. Therefore


Therefore triangles $FLD$ and $EKD$ are similar, so that $\angle LDF = \angle KDE$. The result follows.

Solution 4. [A. Murali] Suppose that $\angle FDH = \alpha$ and $\angle HDE = \beta$. By the Law of Sines,

$$\frac{|AF|}{\sin \alpha} = \frac{|AD|}{\sin \angle AFD}$$

and

$$\frac{|AE|}{\sin \beta} = \frac{|AD|}{\sin \angle AED}.$$
Therefore \[
\frac{\sin \alpha}{\sin \beta} = \frac{|AF|}{|AE|} \cdot \frac{\sin \angle AFD}{\sin \angle AED} = \frac{|AF|}{|AE|} \cdot \frac{\sin \angle BFD}{\sin \angle CED}.
\]

Since \[
\frac{|BD|}{\sin \angle BFD} = \frac{|BF|}{\sin \angle BDF} = \frac{|BF|}{\cos \alpha},
\]
and \[
\frac{|CD|}{\sin \angle CED} = \frac{|CE|}{\cos \beta},
\]
it follows, using Ceva’s theorem, that
\[
\frac{\sin \alpha}{\sin \beta} = \frac{|AF|}{|AE|} \cdot \frac{|BD|}{|BF|} \cdot \frac{|CE|}{|CD|} \cdot \frac{\cos \alpha}{\cos \beta} = \frac{|AF|}{|AE|} \cdot \frac{|BD|}{|BF|} \cdot \frac{|CE|}{|CD|} \cdot \frac{\cos \alpha}{\cos \beta}.
\]
Therefore \(\tan \alpha = \tan \beta\) and the desired result follows.

**Comment.** It was intended that \(D\) be an interior point of \(BC\). However, in the case that either \(B\) or \(C\) is obtuse, the result can be adapted.

569. Let \(A, W, B, U, C, V\) be six points in this order on a circle such that \(AU, BV\) and \(CW\) all intersect in the common point \(P\) at angles of 60°. Prove that
\[
|PA| + |PB| + |PC| = |PU| + |PV| + |PW|.
\]

**Solution 1.** [A. Abdi] We first recall the result: Suppose that \(DEF\) is an equilateral triangle and that \(G\) is a point on the minor arc \(EF\) of the circumcircle of \(DEF\). Then \(|DG| = |EG| + |FG|\). (Select \(H\) on \(DG\) so that \(EH = EG\). Since \(\angle EGH = 60^\circ\), triangle \(EGH\) is equilateral. It can be shown that triangle \(DEH\) and \(FEG\) are congruent (SAS), so that \(|DG| = |DH| + |HG| = |FE| + |EG|).)

Let \(O\) be the centre of the circle and let \(K, M, N\) be respective feet of the perpendiculars from \(O\) to \(AU, BV, CW\). Wolog, let \(K\) be between \(P\) and \(A, M\) between \(P\) and \(V\) and \(N\) be between \(P\) and \(C\). Since triangles \(PKO, PMO\) and \(PNO\) are right with hypotenuse \(PO\), the points \(O, P, K, M, N\) are all equidistant from the midpoint of \(OP\) and so are concyclic.

\(P\) and \(M\) lie on opposite arcs \(KN\) so \(\angle NKM = 180^\circ - \angle NPK = 180^\circ - \angle CPA = 60^\circ\). Also \(\angle NKM = \angle NPM = 60^\circ\) and \(\angle KNM = \angle KPM = 60^\circ\), so that triangle \(KMN\) is equilateral and \(|PM| = |PK| + |PN|\).

Hence
\[
(|AP| + |BP| + |CP|) - (|UP| + |VP| + |WP|) = (|AK| + |PK| + |BM| - |PM| + |CN| + |PN|)
- (|UK| - |PK| + |VM| + |PM| + |WN| - |PN|)
= (|AK| - |UK|) + (|BM| - |VM|) + (|WN| - |PN|) + 2(|PK| - |PM| + |PN|) = 0.
\]

**Comment.** Several solvers tried the strategy of comparing the equation for two related positions, either with the situation where the second position put \(P\) at the centre of the circle, where the result is obvious, or moved \(P\) along one of the lines, say \(UA\) to a new position. In both cases, the fact that the difference in the lengths of two parallel chords was split evenly to the two half chords played a role, as did the perpendiculars to the chords for one position of \(P\) from the other position of \(P\).

**Solution 2.** [P.J. Zhao] Construct equilateral triangles \(BCD\) and \(VWT\) external to \(P\). Then \(PBDC\) and \(PWTV\) are concyclic quadrilaterals so that \(\angle DPC = \angle DBC = 60^\circ = \angle UPC\) and \(\angle TPV = \angle TWV = 60^\circ = \angle APV\). Therefore, the points \(D, U, P, A, T\) are collinear.
Similarly, implies that D, U, P, A, T triangle and equilateral triangle not on the common side passes through it. In the present situation, this external equilateral triangle is erected on one side of the triangle, the line joining the vertices of the given solution 2. Observe that P from which the rays to the vertices meet at an angle of 120° through D̸̸̸, then Thus, Similarly, triangles DOT is isosceles and so OD = OT. Also OU = OA and OUT = OAD. Therefore triangles DAO and TUO are congruent (ASA) and so DA = UT. Hence |

\[ |PA| + |PB| + |PC| = |PU| + |PV| + |PW| \]

Solution 3. [J. Zung] Construct the equilateral triangles BCD and WVT and adopt the notation of Solution 2. Observe that P is the Fermat point of both triangles ABC and UVW; this is the point that minimizes the sum of the distances from P to the vertices of the triangle and is characterized as that point from which the rays to the vertices meet at an angle of 120°. This point has the property, that when an external equilateral triangle is erected on one side of the triangle, the line joining the vertices of the given triangle and equilateral triangle not on the common side passes through it. In the present situation, this implies that D, U, P, A, T are collinear.

Consider the rotation with centre D through an angle of 60° that takes B → C, C → E, P → Q. Then

\[ \angle QCP = \angle QCE + \angle DCE + \angle DBC + \angle DCP \]

\[ = \angle PBC + 60° + 60° + \angle BCP = 180° \]

Thus, Q, C, P are collinear. Since \( \angle PDQ = 60° \), triangle PDQ is equilateral, so that |PQ| = |PD|. Therefore

\[ |PA| + |PB| + |PC| = |PA| + |CQ| + |PC| \]

\[ = |PA| + |PQ| = |PA| + |DP| = |DA| \]

Similarly, |PU| + |PV| + |PT| = |UT|.

Let O be the centre of the circle. Since B, W, V, C are concyclic, \( \angle BCW = \angle BVW \). Since \( \angle BDC + \angle BPC = 180° \), then B, D, C, P are concyclic and \( \angle BDP = \angle BCD \). Since the right bisector of BC passes through D and O, \( \angle BDO = 30° \). Hence

\[ \angle ODP = 30° - \angle BDP = 30° - \angle BCP = 30° - \angle BCW \]

Similarly, \( \angle OTP = 30° - \angle BVW \). Therefore \( \angle ODP = \angle OTP \), triangle ODT is isosceles and so DF = FT, where F is the foot of the perpendicular from O to UA. Since, also, FU = FA, it follows that

\[ |DA| = |DF| + |FA| = |FT| + |FU| = |UT| \]

and the desired result obtains.

Solution 4. [P. Wen] Let the centre of the circle be at the origin, the coordinates of P be (p, q) and the respective lengths of PA, PB, PC, PU, PV, PW be a, b, c, u, v, w. Take UA to be parallel to the x-axis. Then

\[ A \sim (p + a, q) \quad B \sim (p - b/2, q + b\sqrt{3}/2) \quad C = (p - c/2, q - c\sqrt{3}/2) \]
\[ U \sim (p - u, q) \quad V \sim (p + v/2, q - v\sqrt{3}/2) \quad W = (p + w/2, q + w\sqrt{3}/2) \, . \]

Since \( AO = UO \),
\[(p + a)^2 + q^2 = (p - u)^2 + q^2 \implies a^2 + 2ap = u^2 - 2up \]
\[= 0 = (a + u)(a - u + 2p) \implies u = a + 2p \, . \]

Since \( BO = VO \),
\[(p - b/2)^2 + (q + b\sqrt{3}/2)^2 = (p + v/2)^2 + (q - v\sqrt{3}/2)^2 \]
\[\implies b^2 - b(p - q\sqrt{3}) = v^2 + v(p - q\sqrt{3}) \]
\[= 0 = (b + v)(b - v - p + q\sqrt{3}) \implies v = b - p + q\sqrt{3} \, . \]

Since \( CO = WO \),
\[c^2 - c(p + q\sqrt{3}) = w^2 + w(p + q\sqrt{3}) \implies w = c - p - q\sqrt{3} \, . \]

Therefore \( u + v + w = a + b + c \).

Solution 5. Let \(|PA| = a, |PB| = b, |PC| = c, |PU| = u, |PV| = v, |PW| = w\). Let \( r \) be the radius and \( O \) the centre of the circle. Suppose that \(|OP| = d\). Let \( A, W, B \) be on one side of \( OP \) and \( U, C, V \) be on the other side.

Let \( \angle APO = \alpha \leq 60^\circ \). Then \( \angle WPO = \alpha + 60^\circ \), \( \angle BPO = \alpha + 120^\circ \), \( \angle UPO = 180^\circ - \alpha \), \( \angle CPO = 120^\circ - \alpha \), \( \angle VPO = 60^\circ - \alpha \).

Using the Law of Cosines, we obtain that
\[
r^2 = a^2 + d^2 - 2ad \cos \alpha \\
= w^2 + d^2 - 2wd \cos(\alpha + 60^\circ) \\
= b^2 + d^2 - 2bd \cos(\alpha + 120^\circ) \\
= u^2 + d^2 - 2ud \cos(180^\circ - \alpha) = u^2 + d^2 + 2bd \cos \alpha \\
= c^2 + d^2 - 2cd \cos(120^\circ - \alpha) = c^2 + d^2 + 2cd \cos(\alpha + 60^\circ) \\
= v^2 + d^2 - 2vd \cos(60^\circ - \alpha) = v^2 + d^2 + 2vd \cos(\alpha + 120^\circ) \, .
\]

Each of these equations is a quadratic of the form
\[
x^2 - (2d \cos \theta)x + (d^2 - r^2) = 0 \, .
\]
It has one positive and one non-positive root. Since \( r^2 - d^2 \sin^2 \theta \geq d^2 \cos^2 \theta \), the positive root is
\[
\frac{2d \cos \theta + \sqrt{4d^2 \cos^2 \theta - 4d^2 + 4r^2}}{2} = d \cos \theta + \sqrt{r^2 - d^2 \sin^2 \theta} \, .
\]

Hence,
\[
a = d \cos \alpha + \sqrt{r^2 - d^2 \sin^2 \alpha} \, ;
\]
\[
b = d \cos(\alpha + 120^\circ) + \sqrt{r^2 - d^2 \sin^2(\alpha + 120^\circ)} \, ;
\]
\[
c = -d \cos(\alpha + 60^\circ) + \sqrt{r^2 - d^2 \sin^2(\alpha + 60^\circ)} \, ;
\]
\[
u = -d \cos \alpha + \sqrt{r^2 - d^2 \sin^2 \alpha} \, ;
\]
\[
w = d \cos(\alpha + 120^\circ) + \sqrt{r^2 - d^2 \sin^2(\alpha + 120^\circ)} \, ;
\]

therefore
\[
a + b + c - (u + v + w) = 2d[\cos \alpha + \cos(\alpha + 120^\circ) - \cos(\alpha + 60^\circ)] \\
= 2d[\cos \alpha(1 + \cos 120^\circ - \cos 60^\circ) - \sin \alpha(\sin 120^\circ - \sin 60^\circ)] = 0 \, ,
\]
as desired.