Solutions for December

584. Let \( n \) be an integer exceeding 2 and suppose that \( x_1, x_2, \cdots, x_n \) are real numbers for which \( \sum_{i=1}^{n} x_i = 0 \) and \( \sum_{i=1}^{n} x_i^2 = n \). Prove that there are two numbers among the \( x_i \) whose product does not exceed \(-1\).

Solution. We can suppose that the \( x_i \) are ordered in increasing sequence and that there is a positive integer \( k \) with \( x_1 \leq x_2 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n \). Then, noting that \(-x_1 \geq 0\), we have that

\[
\sum_{i=1}^{k} x_i^2 \leq \sum_{i=1}^{k} x_1 x_i = -x_1(x_{k+1} + x_{k+2} + \cdots + x_n) \leq -(n-k)x_1 x_n
\]

and

\[
\sum_{i=k+1}^{n} x_i^2 \leq \sum_{i=k+1}^{n} x_n x_i = -x_n(x_1 + x_2 + \cdots + x_k) \leq -k x_1 x_n.
\]

Finally, \( n = \sum_{i=1}^{n} x_i^2 \leq -n x_1 x_n \); thus \( x_1 x_n \leq -1 \).

585. Calculate the number

\[
a = |\sqrt{n-1} + \sqrt{n} + \sqrt{n+1}|^2,
\]

where \([x]\) denotes the largest integer than does not exceed \( x \) and \( n \) is a positive integer exceeding 1.

Solution. It does not appear that there is a neat expression for this. One can obtain without too much trouble the inequality

\[
3\sqrt{n-1} < \sqrt{n-1} + \sqrt{n} + \sqrt{n+1} < 3\sqrt{n},
\]

from which we can find that when \( k^2 + 1 \leq n \leq k^2 + (2k/3) \), then \( \sqrt{n} = 3k \), when \( k^2 + (2k/3) + (10/9) < n \leq k^2 + (4k/3) + (1/3) \), then \( \sqrt{n} = 3k + 1 \), and when \( k^2 + 4k + (13/9) < n \leq (k + 1)^2 \), then \( \sqrt{n} = 3k + 2 \). However, this leaves the difficulty of getting the right expression for the gaps between the various ranges of \( n \).

586. The function defined on the set \( \mathbb{C}^* \) of all nonzero complex numbers satisfies the equation

\[
f(z)f(iz) = z^2,
\]

for all \( z \in \mathbb{C}^* \). Prove that the function \( f(z) \) is odd, i.e., \( f(-z) = -f(z) \) for all \( z \in \mathbb{C}^* \). Give an example of a function that satisfies this condition.

Solution. Note that \( f(z) \neq 0 \) for all \( z \in \mathbb{C}^* \). Replacing \( z \) by \( iz \) leads to \( f(i z)f(-z) = -z^2 \), from which we have that

\[
f(z)f(iz) + f(iz)f(-z) = 0 \implies f(z) + f(-z) = 0.
\]

Therefore the function is odd.

An example is given by \( f(z) = (-1 + i)z/\sqrt{2} \).

587. Solve the equation

\[
\tan 2x \tan \left(2x + \frac{\pi}{3}\right) \tan \left(2x + \frac{2\pi}{3}\right) = \sqrt{3}.
\]

Solution. Using the standard trigonometric identities for \( \sin A \sin B \), \( \cos A \cos B \), \( \cos 2A \) and \( \sin 2A \), we
have that

\[
\sqrt{3} = \tan 2x \left( \frac{\sin(2x + (\pi/3)) \sin(2x + (2\pi/3))}{\cos(2x + (\pi/3)) \cos(2x + (2\pi/3))} \right)
\]

\[
= \tan 2x \left( \frac{\cos(\pi/3) - \cos(4x + \pi)}{\cos(\pi/3) + \cos(4x + \pi)} \right)
\]

\[
= \tan 2x \left( \frac{1 + 2 \cos 4x}{1 - 2 \cos 4x} \right) = \tan 2x \left( \frac{1 + 2(2 \cos^2 2x - 1)}{1 - 2(1 - 2 \sin^2 2x)} \right)
\]

\[
= \left( \frac{\sin 2x}{\cos 2x} \right) \left( \frac{4 \cos^2 2x - 1}{4 \sin^2 2x - 1} \right) = \frac{2 \sin 4x \cos 2x - \sin 2x}{2 \sin 4x \sin 2x - \cos 2x}
\]

\[
= \frac{\sin 6x + \sin 2x - \sin 2x}{\cos 2x - \cos 6x - \cos 2x} = -\cos 6x
\]

Therefore \( x = -10^\circ + k \cdot 30^\circ \) for some integer \( k \).

588. Let the function \( f(x) \) be defined for \( 0 \leq x \leq \pi/3 \) by

\[
f(x) = \sec \left( \frac{\pi}{6} - x \right) + \sec \left( \frac{\pi}{6} + x \right).
\]

Determine the set of values (its image or range) assumed by the function.

**Solution.** Making use of the inequality \((1/a) + (1/b) \geq 2/\sqrt{ab}\) for \( a, b > 0 \), we find that

\[
f(x) \geq \frac{2}{\sqrt{\cos((\pi/6) - x) \cos((\pi/6) + x)}} \geq \frac{2}{\sqrt{(1/4) + ((\cos 2x)/2)}}.
\]

Since \( 0 \leq x \leq \pi/3 \) implies that \(-\frac{1}{2} \leq \cos 2x \leq 1\), it follows that

\[
0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2},
\]

and

\[
f(x) \geq \frac{4}{\sqrt{3}}.
\]

Since \( f(x) \) is continuous, \( f(0) = 4/\sqrt{3} \) and \( f(x) \) grows without bound when \( x \) approaches \( \pi/3 \), the image of \( f \) on \([0, \pi/3]\) is \([4/\sqrt{3}, \infty)\).

589. In a circle, \( A \) is a variable point and \( B \) and \( C \) are fixed points. The internal bisector of the angle \( BAC \) intersects the circle at \( D \) and the line \( BC \) at \( G \); the external bisector of the angle \( BAC \) intersects the circle at \( E \) and the line \( BC \) at \( F \). Find the locus of the intersection of the lines \( DF \) and \( EG \).

**Solution.** Suppose without loss of generality that \( AB > AC \). If \( M \) is the midpoint of \( BC \), since \( BG : GC = AB : AC, BG > GC \) so that \( G \) lies between \( M \) and \( C \) and \( A \) lies between \( E \) and \( F \). Let \( P \) be the intersection of \( DF \) and \( EG \).

Observe that \( D \) is the midpoint of the arc \( BC \) and that \( AD \perp EF \). Therefore \( DA \) is an altitude of triangle \( DEF \) and \( DE \) is a diameter of the circle. Therefore \( DE \) must pass through \( M \), and so \( FM \perp DE \), i.e., \( FM \) is an altitude of triangle \( DEF \). The intersection of these two altitudes, \( G \), is the orthocentre of triangle \( ABC \) and so \( EG \perp DF \). Thus, \( \angle EPD = 90^\circ \), so that \( P \) must lie on the given circle.

Conversely, let \( P \) be a point on the given circle. Wolog, we may assume that \( P \) lies between \( D \), the midpoint of arc \( BC \) and \( C \). Let \( DE \) be the diameter of the circle that right bisects \( BC \). Suppose that \( DP \) produced intersects \( BC \) produced at \( F \) and that \( EF \) intersects the circle at \( A \). This is the point \( A \) that produced the point \( P \) as described in the problem. Thus, the locus is indeed the given circle with the exception of the points \( B \) and \( C \).
590. Let $SABC$ be a regular tetrahedron. The points $M, N, P$ belong to the edges $SA, SB$ and $SC$ respectively such that $MN = NP = PM$. Prove that the planes $MNP$ and $ABC$ are parallel.

Solution. Let $|SM| = a$, $|SN| = b$ and $|SP| = c$. From the Law of Cosines, we have that $|MN|^2 = a^2 + b^2 - ab$, etc., whence $a^2 + b^2 - ab = b^2 + c^2 - bc = c^2 + a^2 - ac = 0$. This implies that $a = b = c$ [prove it], so that $SM : SA = SN : SB = SP : SC$ and the result follows.