

PROBLEMS FOR OCTOBER

Please send your solutions to

Edward Barbeau
Department of Mathematics
University of Toronto
40 St. George Street
Toronto, ON M5S 2E4

no later than November 21, 2008. Electronic files can be sent to barbeau@math.utoronto.ca. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download. It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are infinitely many solutions to the equation for which the condition is satisfied and x, y, z are all positive.

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB , for which the segments AU, BV, CW intersect in a common point O . Prove that

$$\frac{|OU|}{|AU|} + \frac{|OV|}{|BV|} + \frac{|OW|}{|CW|} = 1,$$

and

$$\frac{|AO|}{|OU|} \cdot \frac{|BO|}{|OV|} \cdot \frac{|CO|}{|OW|} = \frac{|AO|}{|OU|} + \frac{|BO|}{|OV|} + \frac{|CO|}{|OW|} + 2.$$

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA , construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD . Prove that one of the triangles MON and LOK is a 90° rotation of the other around O .

What happens when $ABCD$ is a parallelogram?

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60^\circ$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n . For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n - 1)$.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b .
- (b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d