PROBLEMS FOR APRIL

Please send your solution to
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no later than April 30, 2008. Electronic files can be sent to rosumihai@yahoo.ca.

It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes. \( \lfloor x \rfloor \), the floor of \( x \), is the largest integer \( n \) that does not exceed \( x \), i.e., that integer \( n \) for which \( n \leq x < n + 1 \). \( \{ x \} \), the fractional part of \( x \), is equal to \( x - \lfloor x \rfloor \). The notation \( [PQR] \) denotes the area of the triangle \( PQR \). A geometric progression is a sequence for which the ratio of two successive terms is always the same; its \( n \)th term has the general form \( ar^{n-1} \). A truncated pyramid is a pyramid with a similar pyramid on a base parallel to the base of the first pyramid removed. A polyhedron is inscribed in a sphere if each of its vertices lies on the surface of the sphere.

542. Solve the system of equations

\[
\begin{align*}
\lfloor x \rfloor + 3\{y\} &= 3.9, \\
\{x\} + 3\{y\} &= 3.4.
\end{align*}
\]

543. Let \( a > 0 \) and \( b \) be real parameters, and suppose that \( f \) is a function taking the set of reals to itself for which

\[
f(a^3x^3 + 3a^2bx^2 + 3ab^2x) \leq x \leq a^3f(x)^3 + 3a^2bf(x)^2 + 3ab^2f(x),
\]

for all real \( x \). Prove that \( f \) is a one-one function that takes the set of real numbers onto itself (i.e., \( f \) is a bijection).

544. Define the real sequences \( \{a_n : n \geq 1\} \) and \( \{b_n : n \geq 1\} \) by \( a_1 = 1 \), \( a_{n+1} = 5a_n + 4 \) and \( 5b_n = a_{n+1} \) for \( n \geq 1 \).

(a) Determine \( \{a_n\} \) as a function of \( n \).

(b) Prove that \( \{b_n : n \geq 1\} \) is a geometric progression and evaluate the sum

\[
S = \frac{\sqrt{b_1}}{\sqrt{b_2} - \sqrt{b_1}} + \frac{\sqrt{b_2}}{\sqrt{b_3} - \sqrt{b_2}} + \cdots + \frac{\sqrt{b_n}}{\sqrt{b_{n+1}} - \sqrt{b_n}}.
\]

545. Suppose that \( x \) and \( y \) are real numbers for which \( x^3 + 3x^2 + 4x + 5 = 0 \) and \( y^3 - 3y^2 + 4y - 5 = 0 \). Determine \( (x + y)^{2008} \).

546. Let \( a, a_1, a_2, \ldots, a_n \) be a set of positive real numbers for which

\[
a_1 + a_2 + \cdots + a_n = a
\]

and

\[
\sum_{k=1}^{n} \frac{1}{a - a_k} = \frac{n + 1}{a}.
\]

Prove that

\[
\sum_{k=1}^{n} \frac{a_k}{a - a_k} = 1.
\]
547. Let $A, B, C, D$ be four points on a circle, and let $E$ be the fourth point of the parallelogram with vertices $A, B, C$. Let $AD$ and $BC$ intersect at $M$, $AB$ and $DC$ intersect at $N$, and $EC$ and $MN$ intersect at $F$. Prove that the quadrilateral $DENF$ is concyclic.

548. In a sphere of radius $R$ is inscribed a regular hexagonal truncated pyramid whose big base is inscribed in a great circle of the sphere (i.e., a whose centre is the centre of the sphere). The length of the side of the big base is three times the length of the side of a small base. Find the volume of the truncated pyramid as a function of $R$. 