Solutions for November

521. On a $8 \times 8$ chessboard, either $+1$ or $-1$ is written in each square cell. Let $A_k$ be the product of all the numbers in the $k$th row, and $B_k$ the product of all the numbers in the $k$th column of the board ($k = 1, 2, \ldots, 8$). Prove that the number

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8$$

is a multiple of 4.

Solution 1. It is clear that the value of each $A_k$ and $B_k$ is $+1$ or $-1$. Assume that $p$ of the eight $A_k$ have the value 1 and $8 - p$ have the value $-1$. Similarly, suppose that $q$ of the eight $B_k$ have the value 1 and $8 - q$ have the value $-1$. The each product is the product of all the entries, the products $A_1A_2\cdots A_8$ and $B_1B_2\cdots B_8$ are equal, so that $(-1)^{8-p} = (-1)^{8-q}$ and $p$ and $q$ have the same parity. We have that

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8 = p + (8 - p)(-1) + q + (8 - q)(-1) = 2(p + q) - 16.$$

Since $p + q$ is even, both terms on the right are divisible by 4 and the result follows.

Solution 2. The proof is by induction on the number of negative entries in the square array. If all of the entries are equal to $+1$, then the sum in the problem is equal to 16, which is divisible by 4. Let $n$ be a positive integer, and suppose that the result holds when there are $n$ entries equal to $-1$. Let an array $U$ be given for which there are exactly $n$ entries equal to $-1$. Let $V$ be the array obtained from $U$ by changing exactly one of the entries $-1$ to $+1$, say the entry in the $r$th row and $s$th column. Then the numbers $A_i$ and $B_j$ are the same for both arrays when $i \neq r$ and $j \neq s$.

If $A_k$ and $B_k$ denote the row and column products for the matrix $V$, then the sum of the problem for the array $U$ is obtained from that for the matrix $V$ by the addition of $-2A_r - 2B_s = -2(A_r + B_s)$. Since $(A_r, B_s)$ has one of the values $(+1, +1), (+1, -1), (-1, +1), (-1, -1)$, it follows that the sum is altered by a multiple of 4. Since by the induction hypothesis, the sum for $V$ is divisible by 4, then so also must be the sum for $U$.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always “between” the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio $\sqrt{5} : \sqrt{2} : 1$.

Solution 1. (a) Let $ABC$ be a triangle and let $P$, $K$ and $M$ be the respective intersections of the altitude, angle bisector and median from $A$ in the side $BC$. Suppose, wolog, $AB < AC$. Then (by Pythagoras’ Theorem, for example), $BP < CP$, so that $\angle BAP < \angle CAP$ and the bisector $AK$ of angle $A$ falls within the angle $CAP$. Hence, $BP < BK$. Since $KB : KC = AB : AC, KB < KC$ and the midpoint $M$ of $BC$ must lie in the segment $KC$. The result follows.

(b) Use the same notation as in (a). We may assume that $|AP| = 1, |AK| = \sqrt{2}$ and $|AM| = \sqrt{5}$. We first note that the altitude from $A$ must lie outside of the triangle. Suppose, on the contrary, that $P$ lies on the side $BC$. By Pythagoras’ Theorem, we have that $|PK| = 1$, so that $\angle PAK = 45^\circ$. Then

$$\angle BAP + 45^\circ = \angle BAK = \angle CAK = \angle CAP - 45^\circ,$$

so that

$$\angle CAP = \angle BAP + 90^\circ > 90^\circ,$$

which is impossible.

Hence $P$ must lie on $CB$ produced and $B$ lies in the segment $PK$. Let $|PB| = x$, so that $|BK| = 1 - x, |PM| = 2$ (by Pythagoras’ Theorem), $|KM| = 1, |MC| = 2 - x$ and $|PC| = 4 - x$. We have that

$$45^\circ - \angle PAB = \angle BAK = \angle CAK = \angle PAC - 45^\circ,$$

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so that $\angle PAC = 90^\circ - \angle PAB$ and

$$4 - x = \tan \angle PAC = \cot \angle PAB = \frac{1}{x}.$$  

Thus, $x^2 - 4x + 1 = 0$ and $x = 2 - \sqrt{3}$. We reject the larger root as it would be the reciprocal of the smaller and so it would be the tangent of $\angle PAC$ which is larger than $\angle PAB$.

Therefore, $\tan \angle PAB = 2 - \sqrt{3}$ and so, from the double angle formula, $\tan 2\angle PAB = 1/\sqrt{3}$ Thus, $\angle PAB = 15^\circ$, $\angle PAC = 75^\circ$ and $\angle BAC = 60^\circ$. Since $\angle PBA = 75^\circ$, it follows that $\angle ABC = 105^\circ$ and $\angle BCA = 15^\circ$.

(It can also be checked that $|AB| = 2\sqrt{2 - \sqrt{3}}$, $|AC| = 2\sqrt{2 + \sqrt{3}}$ and $|BC| = 2\sqrt{3}$.

**Solution 2.** (a) can be established as before. For (b), assume wolog that $AC > AB$. We first establish that $\angle ABC$ is obtuse. Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle ACB = \gamma$. Since $\beta > \gamma$,

$$\angle AKC = \beta + \alpha/2 > \gamma + \alpha/2 = \angle AKB,$$

so that $\angle AKB < 90^\circ$ (which agrees with $\angle AKP = 45^\circ$) and $\angle AKC < 90^\circ$ (more precisely, $\angle AKC = 135^\circ$). Hence $\beta + \alpha/2 = \angle AKC = 135^\circ$, so that $180^\circ - \beta - \gamma = \alpha = 270^\circ - 2\beta$ and $\beta = \gamma + 90^\circ > 90^\circ$.

By Pythagoras’s theorem, $|PK| = 1$, $|PM| = 2$ and $|KM| = 1$. Let $|PB| = x$, so that $|BK| = 1 - x$, $|BM| = 2 - x$, $|BC| = 2|BM| = 4 - 2x$, and $|PC| = 4 - x$.

The triangles $ACP$ and $BAP$ are similar since both are right and

$$\angle PAB = \angle ABC - 90^\circ = \beta - 90^\circ = \gamma = \angle ACP.$$  

Therefore $AP : PC = BP : AP$, or, equivalently, $1 : (4 - x) = x : 1$. Therefore, $x$ is the smaller of the roots of $x^2 - 4x + 1 = 0$, namely $2 - \sqrt{3}$.

Thus, $\tan \angle PAB = 2 - \sqrt{3}$, so that $\angle PAB = 15^\circ$. (One way to check this is to use the double angle formula to find the tangent of $15^\circ$.) Therefore, $\gamma = \angle ACB = \angle PAB = 15^\circ$, $\beta = \angle ABC = \gamma + 90^\circ = 105^\circ$ and $\alpha = \angle BAC = 60^\circ$.

**Solution 3.** [J. Schneider] Wolog, let $\angle B > \angle C$. We use the notation of the first solution. If $B$ is obtuse, then $B$ lies between $P$ and $K$. Since $AB < AC$, $BK : KC = AB : AC$, so that $BK < KC$ and $M$ lies between $K$ and $C$.

Let the angle at $B$ be acute. Then $BP : PC = \tan C : \tan B$, $BK : KC = c : b = \sin C : \sin B$ and $BM : MC = 1 : 1$. Since $\sin C < \sin B$ and $\cos C > \cos B$,

$$\frac{\tan C}{\tan B} = \frac{\cos B}{\cos C} \cdot \frac{\sin C}{\sin B} < \frac{\sin C}{\sin B} < 1,$$

and the result follows.

(b) Let $x = |MC|$ and coordinatize the situation by $A \sim (0, 1), B \sim (0, 0), K \sim (1, 0), M \sim (2, 0), C \sim (2 + x, 0)$ and $B \sim (2 - x, 0)$. The proportion $AB^2 : AC^2 = AK^2 : KC^2$ leads to the equation

$$\frac{(x + 1)^2}{(x - 1)^2} = \frac{x^2 + 4x + 5}{x^2 - 4x + 5}$$

which simplifies to $x(x^2 - 3) = 0$. Since $\text{vert} AK < |AC|$, we reject $x = -\sqrt{3}$. Hence $x = \sqrt{3}$. Note that this places $B$ to the right of the origin and so angle $B$ is obtuse.

Thus $|AB| = 2\sqrt{2 - \sqrt{3}}$, $|AC| = 2\sqrt{2 + \sqrt{3}}$ and $|BC| = 2\sqrt{3}$. Angle $A$ can be identified using the Law of Cosines and the remaining angles from their tangents.
523. Let $ABC$ be an isosceles triangle with $AB = AC$. The segments $BC$ and $AC$ are used as hypotenuses to construct three right triangles $BCM$, $BCN$ and $ACP$. Prove that, if $\angle ACP + \angle BCM + \angle BCN = 90^\circ$, then the triangle $MPN$ is isosceles.

**Solution 1.** Clearly, $M$ and $N$ are points on a circle whose diameter is $BC$. Let $O$ be the midpoint of $BC$ and the centre of this circle, and $Q$ the intersection point of the ray $PO$ and the circle. We have that

$$\angle BCM + \angle BCN = \frac{1}{2}(\text{arc } BM + \text{arc } BN). \quad (1)$$

Observe that, as triangle $ABC$ is isosceles with $O$ the midpoint of its base $BC$, $AO \perp BC$. Therefore, $O$ and $P$ are on the circle with diameter $AC$, so that

$$90^\circ - \angle ACP = \angle PAC = \angle POC = \angle BOQ. \quad (2)$$

We are given that $90^\circ - \angle ACP = \angle BCM + \angle BCN$, so that (1) and (2) yield

$$\text{arc } BQ = \angle BOQ = \frac{1}{2}(\text{arc } BM + \text{arc } BN)$$

or

$$\text{arc } BN + \text{arc } NQ + \text{arc } BQ = \frac{1}{2}(\text{arc } BN + \text{arc } MN + \text{arc } BN)$$

which in turn is equivalent to $\text{arc } NQ = \frac{1}{2}(\text{arc } MN)$. Thus, $Q$ is the midpoint of the arc $MN$, so that $PQ$ is the right bisector of the segment $MN$. The result follows.

**Solution 2.** [J. Schneider] Note that triangle $MPN$ is isosceles with $PM = PN$ if and only if the right bisector of $MN$ passes through $P$.

Let $D$ be the midpoint of $BC$. Since $ABC$ is isosceles, $AD \perp BC$ and $D$ lies on the circle with diameter $AC$. Thus, $APCD$ is concyclic and $\angle ADP = \angle ACP$.

Since $D$ is the centre of the circle with diameter $BC$ that contains $M$ and $N$, $\angle BDN = 2\angle BCN$ and $\angle BDM = 2\angle BCM$. Let $X$ be the midpoint of $MN$. Then $DX$ right bisects $MN$ and bisects angle $MDN$. Hence

$$\angle BDX = \frac{1}{2}(\angle BDM + \angle BDN) = \angle BCN + \angle BCM.$$

Suppose that $\angle BCN + \angle BCM + \angle ACP = 90^\circ$, as hypothesized. Then

$$\angle PDC = 90^\circ - \angle ADP = 90^\circ - \angle ACP = \angle BCM + \angle BCN = \angle BDX.$$

Hence $X, D, P$ are collinear. But $DX$ is the right bisector of $MN$, and so is $DP$. Hence triangle $MPN$ is isosceles.

**Comment.** The above argument applies when all triangles are external to triangle $ABC$. It can be adapted to the other cases.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2 - 10x + 26} + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 41}} = x^4 - 9x^3 + 16x^2 + 15x + 26.$$

**Solution.** Observe that

$$x^4 - 9x^3 + 16x^2 + 15x + 26 = (x^2 + x + 1)(x - 5)^2 + 1.$$
Since \( x^2 + x + 1 = \left( x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \) for all \( x \), the quartic on the right side of the equation is never less than 1 and is equal to 1 if and only if \( x = 5 \).

Since \( x^2 - 10x + 25 + k = (x - 5)^2 + k \) for \( k = 1, 4, 16 \), the left side of the equation is never greater than 1 and is equal to 1 if and only if \( x = 5 \). It follows that \( x = 5 \) is the only solution of the equation.

525. The circle inscribed in the triangle \( ABC \) divides the median from \( A \) into three segments of the same length. If the area of \( ABC \) is \( 6\sqrt{14} \), calculate the lengths of its sides.

**Solution.** Let the median from \( A \) meet the side \( BC \) at \( M \). Let \( a, b, c \) denote the side lengths of \( ABC \) as usual, and let the length of the median \( AM \) be \( 3u \). Suppose that the incircle of triangle \( ABC \) touches sides \( BC, CA, AB \) at \( U, V, W \), respectively. Suppose, wolog, that \( AB < AC \), so that \( U \) lies between \( B \) and \( M \).

By the power of a point, we have that \( |AV|^2 = 2u^2 = |MU|^2 \), so that
\[
(1/2)(b + c - a) = |AV| = |MU| = (1/2)a - (1/2)(a + c - b) = (1/2)(b - c) ,
\]
and \( 8u^2 = b^2 - 2bc + c^2 \). Hence \( b + c - a = b - c \), whence \( a = 2c \) and \( |BM| = |MC| = |AB| = c \). By the Law of Cosines applied to triangles \( ABM \) and \( AMC \), with \( \alpha = \angle AMB \),
\[
c^2 = c^2 + (3u)^2 - 6uc\cos\alpha
\]
and
\[
b^2 = c^2 + (3u)^2 + 6uc\cos\alpha ,
\]
whence
\[
b^2 = c^2 + 18u^2 = (9/4)(b^2 - 2bc + c^2) .
\]
This simplifies to
\[
0 = 5b^2 - 18bc + 13c^2 = (b - c)(5b - 13c) .
\]
Since \( b \neq c \) (otherwise, the median from \( A \) would be the angle bisector of \( A \) and the incircle would touch \( BC \) at \( M \)), we must have \( b = 13c/5 \). Hence \( (a, b, c) = (2c, 13c/5, c) \), the semiperimeter of the triangle is \( 14c/5 \) and the square of its area is \((1/5^4)(14c)(4c)c(9c) = (c^4/5^4)(14)(36) .\) Since we are given that the square of the area is \( (14)(36) \), \( c = 5 \) and the dimensions of the triangle are \( (10, 5, 13) \).

**Comment.** All triangles described in the first sentence of the problem have a common property, in that their sides are in the ratio \( 10 : 5 : 13 \). This is, in fact, the essence of the problem. There are many modifications with the same core idea; for example, instead of giving the area of the triangle, we could give the length of the altitude from \( B \), of the angle bisector from \( C \) or of the median from \( A \). Recall that these last three quantities are given respectively by
\[
h_b = \frac{2}{b} \sqrt{s(s-a)(s-b)(s-c)}
\]
\[
l_c = \frac{2}{a+b} \sqrt{abs(s-c)}
\]
\[
m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}
\]
where \( s = \frac{1}{2}(a + b + c) \) is the semiperimeter of the triangle.

526. For the non-negative numbers \( a, b, c \), prove the inequality
\[
4(a + b + c) \geq 3(a + \sqrt{ab} + \sqrt{abc}) .
\]
When does equality hold?
Solution 1. Equality holds when \( a = b = c = 0 \). The inequality clearly holds when \( a = 0 \) and when \( b = 0 \), so henceforth we will assume that \( ab \neq 0 \). Define the nonnegative numbers \( u \) and \( v \) by
\[
\begin{align*}
  u^2 &= \frac{b}{a} \quad \text{and} \quad v^3 = \frac{bc}{a^2}.
\end{align*}
\]
Dividing the inequality through by \( a \), we see that it is equivalent to
\[
4 \left( 1 + u^2 + \frac{v^3}{u^2} \right) \geq 3(1 + u + v)
\]
or
\[
4(u^2 + u^4 + v^3) \geq 3(u^2 + u^3 + u^2v).
\]

The difference between the two members of the last inequality is
\[
4u^4 - 3u^3 + u^2 + 4v^3 - 3u^2v = u^2(2u - 1)^2 + (2v - u)^2(v + u).
\]

Because of the square terms, it is always nonnegative, and it is equal to zero if and only if \( (u, v) = (1/2, 1/4) \). This is achieved when \( a : b : c = 16 : 4 : 1 \). Therefore, the inequality always holds and equality occurs when \( (a, b, c) = (16t, 4t, t) \) for some nonnegative value of \( t \).

Comment. Since the genesis of the solution is far from obvious, it might be worth commenting on how it was arrived at. It is straightforward to dispose of the cases in which any of the variables vanish, so we may as well suppose that all are positive. We observe that the left and right sides of the inequality are homogeneous of degree 1, so that any scalar multiple of a solution vector is also a solution. Thus, we might as well assume that \( a = 1 \). The next step is to get rid of the radicals, which we can do by assuming the quantity under the square root sign is \( u^2 \) and under the cube root sign is \( v^3 \); it is now a matter of backtracking to define these in terms of \( a, b \) and \( c \). Some manipulation gives an equivalent polynomial inequality in terms of \( u \) and \( v \). We now look at the difference between the two sides and investigate the possibility of getting some representation of this difference in terms of squares and things known to be positive. However, all these machinations can be avoided by a little insight, as we shall see in the next solution.

\( 4u^4 - 3u^3 + u^2 \) is almost a square, so we might as well complete it by subtracting \( u^3 \) and adding it to the rest of the expression to get \((2u^2 - u)^2 + (4v^3 - 3u^2v + u^3)\). We notice that the expression in the second parentheses vanished when \( v = -u \), which makes \( v + u \) a factor of it. The remaining factor turns out to be \((2v - u)^2\) and we are finished.

Solution 2. [J. Schneider] Let \( a = u, b = v/4 \) and \( c = w/16 \). The inequality is equivalent to
\[
4\left( u + \frac{v}{4} + \frac{w}{16} \right) \geq 3\left( u + \frac{1}{2}\sqrt{uvw} + \frac{1}{4}\sqrt[3]{uvw} \right).
\]
Since \( 3u \geq 3u, (3/4)(u + v) \geq (3/2)\sqrt{uv} \) and \((1/4)(u + v + w) \geq 3\sqrt[3]{uvw} \) (the last two by the arithmetic-geometric means inequality), the desired inequality follows. Equality occurs if and only if \( u = v = w \), or \( a = 4b = 16c \).

Solution 3. The left side of the inequality can be rewritten
\[
4(a + b + c) = 3a + \frac{3}{4}(a + 4b) + \frac{1}{4}(a + 4b + 16c).
\]
Using the arithmetic-geometric means inequality, we have that
\[
a + 4b \geq 2\sqrt{a(4b)} = 4\sqrt{ab}
\]
and
\[
a + 4b + 16c \geq 3\sqrt[3]{a(4b)(16c)} = 12\sqrt[3]{abc},
\]

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from which the desired result follows. Equality occurs if and only if $a = 4b = 16c$.

527. Consider the set $A$ of the $2n$–digit natural numbers, with 1 and 2 each occurring $n$ times as a digit, and the set $B$ of the $n$–digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring with equal frequency. Show that $A$ and $B$ contain the same number of elements (i.e., have the same cardinality).

Solution 1. We show that $A$ and $B$ have the same number of elements by pairing off the elements of one set with elements of the other. Suppose that a number in $A$ is given; separate it into $n$ consecutive pairs of digits; these pairs will be one of 11, 12, 21, 22. Observe that, since the digits 1 and 2 occur equally frequently, the pairs 11 and 22 must occur equally frequently. Moving from left to right, we construct an $n$–digit number by replacing each pair 11 by the digit 1, 22 by the digit 2, 12 by the digit 3 and 21 by the digit 4. Thus, for example, the number 1221121121212 corresponds to 32143123. Because the number in $A$ has equally many pairs 11 and 22, the corresponding number will have 1 and 2 occurring equally often and will lie in $B$.

Conversely, given an $n$–digits number in $B$, construct a $2n$–digit number by replacing each 1 by 11, 2 by 22, 3 by 12 and 4 by 21. Because 1 and 2 occur equally often, the pairs 11 and 22 will occur equally often in the resulting number, which will then belong to $A$. Thus the correspondence is one-one and the result follows.

Comment. The number of elements in $A$ is $\binom{2n}{n}$, the number of ways of selecting the places for the $n$ ones. To select numbers in $B$ with $r \leq n/2$ digits equal to 1, we can choose the places for the ones in $\binom{n}{r}$ ways, the places for the twos in $\binom{n}{r}$ ways. This leaves $n - 2r$ places left over, which can be filled with either threes or fours in $2^{n-2r}$ ways. Thus, the number of elements in $B$ is

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{r} \binom{n-r}{r} 2^{2n-2r}.$$