

Problems for December, 2005

Please send your solutions to
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no later than January 20, 2006 It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

416. Let P be a point in the plane.
- Prove that there are three points A, B, C for which $AB = BC$, $\angle ABC = 90^\circ$, $|PA| = 1$, $|PB| = 2$ and $|PC| = 3$.
 - Determine $|AB|$ for the configuration in (a).
 - A rotation of 90° about B takes C to A and P to Q . Determine $\angle APQ$.
417. Show that for each positive integer n , at least one of the five numbers 17^n , 17^{n+1} , 17^{n+2} , 17^{n+3} , 17^{n+4} begins with 1 (at the left) when written to base 10.
418. (a) Show that, for each pair m, n of positive integers, the minimum of $m^{1/n}$ and $n^{1/m}$ does not exceed $3^{1/2}$.
- (b) Show that, for each positive integer n ,

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 \geq n^{1/n} \geq 1.$$

- (c) Determine an integer N for which

$$n^{1/n} \leq 1.00002005$$

whenever $n \geq N$. Justify your answer.

419. Solve the system of equations

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x} = t$$

for x, y, z not all equal. Determine xyz .

420. Two circles intersect at A and B . Let P be a point on one of the circles. Suppose that PA meets the second circle again at C and PB meets the second circle again at D . For what position of P is the length of the segment CD maximum?
421. Let $ABCD$ be a tetrahedron. Prove that

$$|AB| \cdot |CD| + |AC| \cdot |BD| \geq |AD| \cdot |BC|.$$

422. Determine the smallest two positive integers n for which the numbers in the set $\{1, 2, \dots, 3n-1, 3n\}$ can be partitioned into n disjoint triples $\{x, y, z\}$ for which $x + y = 3z$.