Solutions and comments

61. Let $S = 1!2!3!\cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer $k$ for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is $k$ unique? (Optional: Is it possible to find such a number $k$ that exceeds 100?)

**Solution 1.** Note that, for each positive integer $j$, $(2j-1)!(2j)! = [(2j-1)!]^2 \cdot 2j$. Hence

$$S = \prod_{j=1}^{50} [(2j-1)!]^2 [2j] = 2^{50}50! \left( \prod_{j=1}^{50} (2j-1)! \right)^2,$$

from which we see that $k = 50$ is the required number.

We show that $k = 50$ is the only possibility. First, $k$ cannot exceed 100, for otherwise 101! would be a factor of $k!$ but not $S$, and so $S/k!$ would not even be an integer. Let $k \leq 100$. The prime 47 does not divide $k!$ for $k \leq 46$ and divides 50! to the first power. Since $S/50!$ is a square, it evidently divides $S$ to an odd power. So $k \geq 47$ in order to get a quotient divisible by 47 to an even power. The prime 53 divides each $k!$ for $k \geq 53$ to the first power and divides $S/50!$, and so $S$ to an even power. Hence, $k \leq 52$.

The prime 17 divides 50! and $S/50!$, and hence $S$ to an even power, but it divides each of 51! and 52! to the third power. So we cannot have $k = 51$ or 52. Finally, look at the prime 2. Suppose that $2^{2u}$ is the highest power of 2 that divides $S/50!$ and that $2^v$ is the highest power of 2 that divides 50!; then $2^{2u+v}$ is the highest power of 2 that divides $S$. The highest power of 2 that divides 48! and 49! is $2^{v-1}$ and the highest power of 2 that divides 46! and 47! is $2^{v-5}$. From this, we deduce that 2 divides $S/k!$ to an odd power when $47 \leq k \leq 49$. The desired uniqueness of $k$ follows.

**Solution 2.** Let $p$ be a prime exceeding 50. Then $p$ divides each of $m!$ to the first power for $p \leq m \leq 100$, so that $p$ divides $S$ to the even power $100 - (p - 1) = 101 - p$. From this, it follows that if $53 \geq k$, $p$ must divide $S/k!$ to an odd power.

On the other hand, the prime 47 divides each $m!$ with $47 \leq m \leq 93$ to the first power, and each $m!$ with $94 \leq m \leq 100$ to the second power, so that it divides $S$ to the power with exponent $54 + 7 = 61$. Hence, in order that it divide $S/k!$ to an even power, we must make $k$ one of the numbers $47, \ldots, 52$.

By an argument, similar to that used in Solution 1, it can be seen that 2 divides any product of the form $1!2!\cdots(2m-1)!$ to an even power and 100! to the power with exponent $\sum_{j=1}^{100/2} + \sum_{j=1}^{100/4} + \sum_{j=1}^{100/8} + \sum_{j=1}^{100/16} + \sum_{j=1}^{100/32} + \sum_{j=1}^{100/64} = 50 + 25 + 12 + 6 + 3 + 1 = 97$.

Hence, 2 divides $S$ to an odd power. So we need to divide $S$ by $k!$ which 2 divides to an odd power to get a perfect square quotient. This reduces the possibilities for $k$ to 50 or 51. Since

$$S = 2^{99} \cdot 3^{98} \cdot 4^{97} \cdots 99^2 \cdot 100 = (2 \cdot 4 \cdots 50)(2^{49} \cdot 3^{49} \cdot 4^{48} \cdots 99)^2 = 50! \cdot 2^{50} \cdots ^2,$$

$S/50!$ is a square, and so $S/5! = (S/50!) \div (51)$ is not a square. The result follows.

**Solution 3.** As above, $S/(50!)$ is a square. Suppose that $53 \leq k \leq 100$. Then 53 divides $k!/50!$ to the first power, and so $k!/50!$ cannot be square. Hence $S/k! = (S/50!) \div (k!/50!)$ cannot be square. If $k = 51$ or 52, then $k!/50!$ is not square, so $S/k!$ cannot be square. Suppose that $k \leq 46$. Then 47 divides $50!/k!$ to the first power, so that $50!/k!$ is not square and $S/k! = (S/50!) \times (50!/k!)$ cannot be square. If $k = 47, 48$ or 49, then $50!/k!$ is not square and so $S/k!$ is not square. Hence $S/k!$ is square if and only if $k = 50$ when $k \leq 100$.

62. Let $n$ be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

**Solution.** Any prime divisor of $n! + 1$ must be larger than $n$, since all primes not exceeding $n$ divide $n!$. Suppose, if possible, the result fails. Then, the only prime that can divide $n! + 1$ is $n + 1$, so that, for some positive integer $r$ and nonnegative integer $K$,

$$n! + 1 = (n + 1)^r = 1 + rn + Kn^2.$$
This happens, for example, when \( n = 1, 2, 4: \) \( 1! + 1 = 2, 2! + 1 = 3, 4! + 1 = 5^2 \). Note, however, that the desired result does hold for \( n = 3: 3! + 1 = 7 \).

Henceforth, assume that \( n \) exceeds 4. If \( n \) is prime, then \( n + 1 \) is composite, so by our initial comment, all of its prime divisors exceed \( n + 1 \). If \( n \) is composite and square, then \( n! \) is divisible by the four distinct integers \( 1, n, \sqrt{n}, 2\sqrt{n} \), while is \( n \) is composite and nonsquare with a nontrivial divisor \( d \), then \( n! \) is divisible by the four distinct integers \( 1, d, n/d, n \). Thus, \( n! \) is divisible by \( n^2 \). Suppose, if possible, the result fails, so that \( n! + 1 = 1 + rn + Kn^2 \), and \( 1 \equiv 1 + rn \mod{n^2} \). Thus, \( r \) must be divisible by \( n \), and, since it is positive, must exceed \( n \). Hence

\[
(n + 1)^r \geq (n + 1)^n > (n + 1)(n - 1) \cdots 1 > n! + 1,
\]
a contradiction. The desired result follows.

63. Let \( n \) be a positive integer and \( k \) a nonnegative integer. Prove that

\[
n! = (n + k)^n - \binom{n}{1}(n + k - 1)^n + \binom{n}{2}(n + k - 2)^n - \cdots \pm \binom{n}{n} k^n.
\]

**Solution 1.** Recall the Principle of Inclusion-Exclusion: Let \( S \) be a set of \( n \) objects, and let \( P_1, P_2, \ldots, P_m \) be \( m \) properties such that, for each object \( x \in S \) each property \( P_i \), either \( x \) has the property \( P_i \) or \( x \) does not have the property \( P_i \). Let \( f(i, j, \ldots, k) \) denote the number of elements of \( S \) each of which has properties \( P_i, P_j, \ldots, P_k \) (and possibly others as well). Then the number of elements of \( S \) each having none of the properties \( P_1, P_2, \ldots, P_m \) is

\[
n - \sum_{1 \leq i \leq m} f(i) + \sum_{1 \leq i < j \leq m} f(i, j) - \sum_{1 \leq i < j < l \leq m} f(i, j, l) + \cdots + (-1)^m f(1, 2, \ldots, m).
\]

We apply this to the problem at hand. Note that an ordered selection of \( n \) numbers selected from among \( 1, 2, \ldots, n + k \) is a permutation of \( \{1, 2, \ldots, n\} \) if and only if it is constrained to contain each of the numbers \( 1, 2, \ldots, n \). Let \( S \) be the set of all ordered selections, and we say that a selection has property \( P_i \) iff its fails to include at least \( i \) of the numbers \( 1, 2, \ldots, n \) \( (1 \leq i \leq n) \). The number of selections with property \( P_i \) is the product of \( \binom{n}{i} \), the number of ways of choosing the \( i \) numbers not included and \( (n + k - i)^n \), the number of ways of choosing entries for the \( n \) positions from the remaining \( n + k - i \) numbers. The result follows.

**Solution 2.** We begin with a lemma:

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = \begin{cases} 
0 & (0 \leq m \leq n - 1) \\
(-1)^n n! & (m = n). 
\end{cases}
\]

We use the convention that \( 0^0 = 1 \). To prove this, note first that \( i(i-1) \cdots (i-m) = i^{m+1} + b_m i^m + \cdots + b_1 i + b_0 \) for some integers \( b_i \). We use an induction argument on \( m \). The result holds for each positive \( n \) and for \( m = 0 \), as the sum is the expansion of \( (1 - 1)^n \). It also holds for \( n = 1, 2 \) and all relevant \( m \). Fix \( n \geq 3 \). Suppose
that it holds when \( m \) is replaced by \( k \) for \( 0 \leq k \leq m \leq n - 2 \). Then
\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} i^{m+1} = \sum_{i=1}^{n} (-1)^i \binom{n}{i} i(i-1) \cdots (i-m) - \sum_{k=0}^{m} b_k \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k
\]
\[
= \sum_{i=m+1}^{n} (-1)^i \binom{n}{i} i(i-1) \cdots (i-m) - 0
\]
\[
= \sum_{i=m+1}^{n} (-1)^i \frac{n!}{i!(n-i)!} \sum_{j=0}^{m-1} (-1)^{m+i-j} \frac{n!}{(n-m-1-j)!j!}
\]
\[
= \sum_{j=0}^{n-m-1} (-1)^{m+i-j} \frac{n(n-1) \cdots (n-m)(n-m-1)!}{(n-m-1-j)!j!}
\]
\[
= (-1)^{m+1} n(n-1) \cdots (n-m) \sum_{j=0}^{n-m-1} (-1)^j \binom{n-m-1}{j} = 0.
\]

(Note that the \( j = 0 \) term is 1, which is consistent with the 0^0 = 1 convention mentioned earlier.) So \( \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^m = 0 \) for \( 0 \leq m \leq n - 1 \). Now consider the case \( m = n \):
\[
\sum_{i=1}^{n} (-1)^i \binom{n}{i} i^n = \sum_{i=1}^{n} (-1)^i \binom{n}{i} i(i-1) \cdots (i-n+1) - \sum_{k=0}^{n-1} b_k \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k.
\]

Every term in the first sum vanishes except the \( n \)th and each term of the second sum vanishes. Hence \( \sum_{i=1}^{n} (-1)^i \binom{n}{i} i^n = (-1)^n n! \).

Returning to the problem at hand, we see that the right side of the desired equation is equal to
\[
(n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots + (-1)^n \binom{n}{n} (n+k-n)^n
\]
\[
= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i+k)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{n} \binom{n}{j} (n-i)^j k^{n-j}
\]
\[
= \sum_{j=0}^{n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n}{j} (n-i)^j k^{n-j} = \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^j
\]
\[
= \sum_{j=0}^{n} \binom{n}{j} k^{n-j} \sum_{i=0}^{n} (-1)^i (n-i)^j.
\]

When \( 0 \leq j \leq n - 1 \), the sum \( \sum_{i=0}^{n} (-1)^i (n-i)^j = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^j \) vanishes, while, when \( j = n \), it assumes the value \( n! \). Thus, the right side of the given equation is equal to \( \binom{n}{n} k^n n! = n! \) as desired.

Solution 3. Let \( m = n + k \), so that \( m \geq n \), and let the right side of the equation be denoted by \( R \). Then
\[
R = m^n - \binom{n}{1}(m-1)^n + \binom{n}{2}(m-2)^n - \cdots + (-1)^n \binom{n}{n} (m-n)^n
\]
\[
= m^n \left[ \sum_{j=0}^{n} (-1)^j \binom{n}{i} \right] - \frac{m^n}{1!} \sum_{i=1}^{n} (-1)^i \binom{n}{i} + \frac{m^{n-2}}{2!} \sum_{i=1}^{n} (-1)^i i^2 \binom{n}{i} + \cdots
\]
\[
+ (-1)^n \frac{n}{n} \left[ \sum_{i=1}^{n} (-1)^i i^n \binom{n}{i} \right].
\]
Let
\[
f_0(x) = (1 - x)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i
\]
and let
\[
f_k(x) = xDf_{k-1}(x)
\]
for \( k \geq 1 \), where \( Df \) denotes the derivative of a function \( f \). Observe that, from the closed expression for \( f_0(x) \), we can establish by induction that
\[
f_k(x) = \sum_{i=0}^{n} (-1)^i k \binom{n}{i} x^i
\]
so that \( R = \sum_{k=0}^{n} (-1)^k \binom{n}{k} m^{n-k} f_k(1) \).

By induction, we establish that
\[
f_k(x) = (-1)^k n(n - 1) \cdots (n - k + 1)x^k(1 - x)^{n-k} + (1 - x)^{n-k+1} g_k(x)
\]
for some polynomial \( g_k(x) \). This is true for \( k = 1 \) with \( g_1(x) = 0 \). Suppose if holds for \( k = j \). Then
\[
f'_j(x) = (-1)^j n(n - 1) \cdots (n - j + 1)x^{j-1}(1 - x)^{n-j} - (-1)^j n(n - 1) \cdots (n - j + 1)(n - j)x^j(1 - x)^{n-j-1}
- (n - j + 1)(1 - x)^{n-j} g_j(x) + (1 - x)^{n-j+1} g'_j(x),
\]
whence
\[
f_{j+1}(x) = (-1)^{j+1} n(n_1) \cdots (n - j)x^j(1 - x)^{n-(j+1)} + (1 - x)^{n-(j+1)+1} [(-1)^j n(n - 1) \cdots (n - j + 1)x^j
- (n - j + 1)xg_j(x) + x(1 - x)g'_j(x)]
\]
and we obtain the desired representation by induction. Then for \( 1 \leq k \leq n - 1 \), \( f_k(1) = 0 \) while \( f_n(1) = (-1)^n n! \). Hence \( R = (-1)^n f_n(1) = n! \).

64. Let \( M \) be a point in the interior of triangle \( ABC \), and suppose that \( D, E, F \) are respective points on the side \( BC, CA, AB \), which all pass through \( M \). (In technical terms, they are cevians.) Suppose that the areas and the perimeters of the triangles \( BMD, CME, AMF \) are equal. Prove that triangle \( ABC \) must be equilateral.

Solution. [L. Lessard] Let the common area of the triangles \( BMD, CME \) and \( AMF \) be \( a \) and let their common perimeter be \( p \). Let the area and perimeter of \( \triangle AME \) be \( u \) and \( x \) respectively, of \( \triangle MFB \) be \( v \) and \( y \) respectively, and of \( \triangle CMD \) be \( w \) and \( z \) respectively.

By considering pairs of triangles with equal heights, we find that
\[
\frac{AF}{FB} = \frac{a}{v} = \frac{2a + u}{v + a + w} = \frac{a + u}{a + w},
\]
\[
\frac{BD}{DC} = \frac{a}{w} = \frac{2a + v}{u + a + w} = \frac{a + v}{a + w},
\]
\[
\frac{CE}{EA} = \frac{a}{u} = \frac{2a + w}{u + a + v} = \frac{a + w}{a + v}.
\]

From these three sets of equations, we deduce that
\[
\frac{a^3}{uvw} = 1.
\]
\[a^2 + (w - u)a - uv = 0 ,
\]
\[a^2 + (u - w)a - vw = 0 ,
\]
\[a^2 + (v - u)a - uw = 0 ;
\]
whence
\[a^3 = uvw \quad \text{and} \quad 3a^2 = uw + vw + uw .
\]

This means that \(uv, vw, uw\) are three positive numbers whose geometric and arithmetic means are both equal to \(a^2\). Hence \(a^2 = w = vw = uw\), so that \(u = v = w = a\). It follows that \(AF = FB, BD = DC, CE = EA\), so that \(AD, BE, CF\) are medians and \(M\) is the centroid.

Wolog, suppose that \(AB \geq BC \geq CA\). Since \(AB \geq BC\), \(\angle AEB \geq 90^\circ\), and so \(AM \geq MC\). Thus \(x \geq p\). Similarly, \(y \geq p\) and \(p \geq z\).

Consider triangles \(BMD\) and \(AME\). We have \(BD \geq AE, BM \geq AM, ME = \frac{1}{2}BM\) and \(MD = \frac{1}{2}AM\). Therefore
\[p - x = (BD + MD + BM) - (AE + ME + AM) = (BD - AE) + \frac{1}{2}(BM - AM) \geq 0
\]
and so \(p \geq x\). Since also \(x \geq p\), we have that \(p = x\). But this implies that \(AM = MC\), so that \(ME \perp AC\) and \(AB = BC\). Since \(BE\) is now an axis of a reflection which interchanges \(A\) and \(C\), as well as \(F\) and \(D\), it follows that \(p = z\) and \(p = y\) as well. Thus, \(AB = AC\) and \(AC = BC\). Thus, the triangle is equilateral.

65. Suppose that \(XTY\) is a straight line and that \(TU\) and \(TV\) are two rays emanating from \(T\) for which \(\angle XTY = \angle UTV = \angle VTY = 60^\circ\). Suppose that \(P, Q\) and \(R\) are respective points on the rays \(TY, TU\) and \(TV\) for which \(PQ = PR\). Prove that \(\angle QPR = 60^\circ\).

Solution 1. Let \(R\) be a rotation of \(60^\circ\) about \(T\) that takes the ray \(TU\) to \(TV\). Then, if \(R\) transforms \(Q \rightarrow Q'\) and \(P \rightarrow P'\), then \(Q'\) lies on \(TV\) and the line \(Q'P'\) makes an angle of \(60^\circ\) with \(QP\). Because of the rotation, \(\angle P'TP = 60^\circ\) and \(TP' = TP\), whence \(TP'P\) is an equilateral triangle.

Since \(\angle Q'TP = \angle TPP' = 60^\circ\), \(TV \parallel P'P\). Let \(T\) be the translation that takes \(P'\) to \(P\). It takes \(Q'\) to a point \(Q''\) on the ray \(TV\), and \(PQ'' = P'Q' = PQ\). Hence \(Q''\) can be none other than the point \(R\) [why?], and the result follows.

Solution 2. The reflection in the line \(XY\) takes \(P \rightarrow P, Q \rightarrow Q'\) and \(R \rightarrow R'\). Triangles \(PQR'\) and \(P'QR\) are congruent and isosceles, so that \(\angle TQP = \angle TPQ = \angle TRP\) (since \(PQ = PR\)). Hence \(TQRP\) is a concyclic quadrilateral, whence \(\angle QPR = \angle QTR = 60^\circ\).

Solution 3. [S. Niu] Let \(S\) be a point on \(TU\) for which \(SR \parallel XY\); observe that \(\Delta RST\) is equilateral. We first show that \(Q\) lies between \(S\) and \(T\). For, if \(S\) were between \(Q\) and \(T\), then \(\angle PSQ\) would be obtuse and \(PQ > PS > PR\) (since \(\angle PRS > 60^\circ > \angle PSR\) in \(\Delta PRS\)), a contradiction.

The rotation of \(60^\circ\) with centre \(R\) that takes \(S\) onto \(T\) takes ray \(RQ\) onto a ray through \(R\) that intersects \(TY\) in \(M\). Consider triangles \(RSQ\) and \(RTM\). Since \(\angle RST = \angle RTM = 60^\circ, \angle SRQ = 60^\circ - \angle QST = \angle TRM\) and \(SR = TR\), we have that \(\Delta RSQ \equiv \Delta RTM\) and \(RQ = RM\). (ASA) Since \(\angle QRM = 60^\circ, \Delta RQM\) is equilateral and \(RM = RQ\). Hence \(M\) and \(P\) are both equidistant from \(Q\) and \(R\), and so at the intersection of \(TY\) and the right bisector of \(QR\). Thus, \(M = P\) and the result follows.

Solution 4. [H. Pan] Let \(Q'\) and \(R'\) be the respective reflections of \(Q\) and \(R\) with respect to the axis \(XY\). Since \(\angle RT'R' = 120^\circ\) and \(TR = TR', \angle QR'R' = \angle TR'R = 30^\circ\). Since \(Q, R, Q', R'\), lie on a circle with centre \(P\), \(\angle QPR = 2\angle Q'R'R = 60^\circ\), as desired.

Solution 5. [R. Barrington Leigh] Let \(W\) be a point on \(TV\) such that \(\angle WPQ = 60^\circ = \angle WTP\). [Why does such a point \(W\) exist?] Then \(WQTP\) is a concyclic quadrilateral so that \(\angle QWP = 180^\circ - \angle QTP = 60^\circ\) and \(\Delta PWQ\) is equilateral. Hence \(PW = PQ = PR\).
Suppose $W \neq R$. If $R$ is farther away from $T$ than $W$, then $\angle RPT > \angle WPT > \angle WPQ = 60^\circ \Rightarrow \angle TRP = \angle RWP > 60^\circ$, a contradiction. If $W$ is farther away from $T$ than $R$, then $\angle WPT > \angle WPQ = 60^\circ \Rightarrow \angle RWP = \angle WRP > 60^\circ$, again a contradiction. So $R = W$ and the result follows.

**Solution 6.** [M. Holmes] Let the circle through $T, P, Q$ intersect $TV$ in $N$. Then $\angle QNP = 180^\circ - \angle QTP = 60^\circ$. Since $\angle PQN = \angle PNT = 60^\circ$, $\triangle PQN$ is equilateral so that $PN = PQ$. Suppose, if possible, that $R \neq N$. Then $N$ and $R$ are two points on $TV$ equidistant from $P$. Since $\angle PNT < \angle PNQ = 60^\circ$ and $\triangle PNR$ is isosceles, we have that $\angle PNR < 90^\circ$, so $N$ cannot lie between $T$ and $R$, and $\angle PRN = \angle PNR = \angle PNT < 60^\circ$. Since $\angle PNT = 60^\circ$, we conclude that $T$ must lie between $R$ and $N$, which transgresses the condition of the problem. Hence $R$ and $N$ must coincide and the result follows.

**Solution 7.** [P. Cheng] Determine $S$ on $TU$ and $Z$ on $TY$ for which $SR \parallel XY$ and $\angle QZR = 60^\circ$. Observe that $\angle BTS = \angle BRT = 60^\circ$ and $SR = RT$.

Consider triangles $\triangle SRQ$ and $\triangle TRZ$. $\angle SRQ = \angle SRT - \angle QRT = \angle QZR - \angle QRT = \angle TRZ$; $\angle QSR = 60^\circ = \angle ZTR$, so that $\angle SRQ = \angle TRZ$ (ASA).

Hence $RZ = RQ \Rightarrow \triangle RQZ$ is equilateral $\Rightarrow RZ = QZ$ and $\angle RZQ = 60^\circ$. Now, both $P$ and $Z$ lie on the intersection of $TY$ and the right bisector of $QR$, so they must coincide: $P = Z$. The result follows.

**Solution 8.** Let the perpendicular, produced, from $Q$ to $XY$ meet $VT$, produced, in $S$. Then $\angle XTS = \angle VTY = 60^\circ = \angle XTU$, from which is can be deduced that $TX$ right bisects $QS$. Hence $PS = PQ = PR$, so that $Q, R, S$ are all on the same circle with centre $P$.

Since $\angle QTS = 120^\circ$, we have that $\angle SQT = \angle QSR = 30^\circ$, so that $QR$ must subtend an angle of $60^\circ$ at the centre $P$ of the circle. The desired result follows.

**Solution 9.** [A. Siu] Let the right bisector of $QR$ meet the circumcircle of $TQR$ on the same side of $QR$ at $T$ in $S$. Since $\angle QSR = \angle QTR = 60^\circ$ and $QS = QR$, $\angle SQR = \angle SRQ = 60^\circ$. Hence $\angle STQ = 180^\circ - \angle SRQ = 120^\circ$. But $\angle YTQ = 120^\circ$, so $S$ must lie on $TY$. It follows that $S = P$.

**Solution 10.** Assign coordinates with the origin at $T$ and the $x$–axis along $XY$. The the respective coordinates of $Q$ and $R$ have the form $(u, -\sqrt{3}u)$ and $(v, \sqrt{3}v)$ for some real $u$ and $v$. Let the coordinates of $P$ be $(w, 0)$. Then $PQ = PR$ yields that $w = 2(u + v)$. [Exercise: work it out.]

\[
|PQ|^2 - |QR|^2 = (u - w)^2 + 3u^2 - (u - v)^2 - 3(u + v)^2 = 2w^2 - 2uv(u + v) = w^2 - 2uv - 2uw = w^2 - 2(u + v)w = 0.
\]

Hence $PQ = QR = PR$ and $\Delta PQR$ is equilateral. Therefore $\angle QPR = 60^\circ$.

**Solution 11.** [J. Y. Jin] Let $C$ be the circumcircle of $\Delta PQR$. If $T$ lies strictly inside $C$, then $60^\circ = \angle QTR > \angle QPR$ and $60^\circ = \angle PTR > \angle PQR = \angle PRQ$. Thus, all three angle of $\Delta PQR$ would be less than $60^\circ$, which is not possible. Similarly, if $T$ lies strictly outside $C$, then $60^\circ = \angle QTR < \angle QPR$ and $60^\circ = \angle PTR < \angle PQR = \angle PRQ$, so that all three angles of $\Delta PQR$ would exceed $60^\circ$, again not possible. Thus $T$ must be on $C$, whence $\angle QPR = \angle QTR = 60^\circ$.

**Solution 12.** [C. Lau] By the Sine Law,

\[
\frac{\sin \angle TQP}{|TP|} = \frac{\sin 120^\circ}{|PQ|} = \frac{\sin 60^\circ}{|PR|} = \frac{\sin \angle TRP}{|TP|},
\]

whence $\sin \angle TQP = \sin \angle TRP$. Since $\angle QTP$ in triangle $QTP$ is obtuse, $\angle TQP$ is acute.

Suppose, if possible, that $\angle TRP$ is obtuse. Then, in triangle $TPR$, $TP$ would be the longest side, so $PR < TP$. But in triangle $TQP$, $PQ$ is the longest side, so $PQ > TP$, and so $PQ \neq PR$, contrary to hypothesis. Hence $\angle TRP$ is acute. Therefore, $\angle TQP = \angle TRP$. Let $PQ$ and $RT$ intersect in $Z$. Then, $60^\circ = \angle QTZ = 180^\circ - \angle TQP - \angle QZT = 180^\circ - \angle TRP - \angle RZP = \angle QPR$, as desired.
66. (a) Let $ABCD$ be a square and let $E$ be an arbitrary point on the side $CD$. Suppose that $P$ is a point on the diagonal $AC$ for which $EP \perp AC$ and that $Q$ is a point on $AE$ produced for which $CQ \perp AE$. Prove that $B, P, Q$ are collinear.

(b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?

Solution 1. Let $ABCD$ be a rectangle, and let $E, P, Q$ be determined as in the problem. Suppose that $\angle ACD = \angle BDC = \alpha$. Then $\angle PEC = 90^\circ - \alpha$. Because $EPQC$ is concyclic, $\angle PQC = \angle PEC = 90^\circ - \alpha$. Because $ABCQ$ is concyclic, $\angle BQC = \angle BDC = \alpha$. The points $B, P, Q$ are collinear $\iff \angle BQC = \angle PQC \iff \alpha = 90^\circ - \alpha \iff \alpha = 45^\circ \iff ABCD$ is a square.

Solution 2. (a) $EPQC$, with a pair of supplementary opposite angles, is concyclic, so that $\angle CQP = \angle CEP = 180^\circ - \angle EPC = \angle ECP = 45^\circ$. Since $CBAQ$ is concyclic, $\angle CQB = \angle CAB = 45^\circ$. Thus, $\angle CQP = \angle CQB$ so that $Q, P, B$ are collinear.

(b) Suppose that $ABCD$ is a non-square rectangle. Then taking $E = D$ yields a counterexample.

Solution 3. (a) The circle with diameter $AC$ that passes through the vertices of the square also passes through $Q$. Hence $\angle QBC = \angle QAC$. Consider triangles $PBC$ and $EAC$. Since triangles $ABC$ and $EPC$ are both isosceles right triangles, $BC : AC = PC : EC$. Also $\angle BCA = \angle PCE = 45^\circ$. Hence $\triangle PBC \sim \triangle EAC$ (SAS) so that $\angle PBC = \angle EAC = \angle QAC = \angle QBC$. It follows that $P, Q, B$ are collinear.

Solution 4. [S. Niu] Let $ABCD$ be a rectangle and let $E, P, Q$ be determined as in the problem. Let $EP$ be produced to meet $BC$ in $F$. Since $\angle ABF = \angle APF$, the quadrilateral $ABPF$ is concyclic, so that $\angle PBC = \angle PBF = \angle PAF$. Since $ABCQ$ is concyclic, $\angle QBC = \angle QAC = \angle PAE$. Now $B, P, Q$ are collinear

$\iff \angle PBC = \angle QBC \iff \angle PAF = \angle PAE \iff AC$ right bisects $EF$

$\iff \angle ECA = \angle ACB = 45^\circ \iff ABCD$ is a square.

Solution 5. [M. Holmes] (a) Suppose that $BQ$ intersects $AC$ in $R$. Since $ABCQD$ is concyclic, $\angle AQR = \angle AQB = \angle ACB = 45^\circ$, so that $\angle BQC = 45^\circ$. Since $\angle EQR = \angle AQB = \angle ECR = 45^\circ$, $ERCQ$ is concyclic, so that $\angle ERC = 180^\circ - \angle EQC = 90^\circ$. Hence $ER \perp AC$, so that $R = P$ and the result follows.

Solution 6. [L. Hong] (a) Let $QC$ intersect $AB$ in $F$. We apply Menelaus’ Theorem to triangle $AFC$: $B, P, Q$ are collinear if and only if

$$\frac{AB}{BF} \cdot \frac{FQ}{QC} \cdot \frac{CP}{PA} = -1.$$

Let the side length of the square be 1 and the length of $DE$ be $a$. Then $|AB| = 1$. Since $\triangle ADE \sim \triangle FBC$, $AD : DE = BF : BC$, so that $|BF| = |1/a$ and $|FC| = \sqrt{1+a^2}/a$. Since $\triangle ADE \sim \triangle CQE$, $CQ : EC = AD : EA$, so that $|CQ| = (1-a)/\sqrt{1+a^2}$. Hence

$$\frac{|FQ|}{|CQ|} = 1 + \frac{|FC|}{|CQ|} = 1 + \frac{1 + a^2}{a(1-a)} = \frac{1 + a}{a(1-a)}.$$

Since $\triangle ECP$ is right isosceles, $|CP| = (1-a)/\sqrt{2}$ and $|PA| = \sqrt{2} - |CP| = (1+a)/\sqrt{2}$. Hence $|CP|/|PA| = (1-a)/(1+a)$. Multiplying the three ratios together and taking account of the directed segments gives the product $-1$ and yields the result.

Solution 7. (a) Select coordinates so that $A \sim (0,1)$, $B \sim (0,0)$, $C \sim (1,0)$, $D \sim (1,1)$ and $E \sim (1,t)$ for some $t$ with $0 \leq t \leq 1$. It is straightforward to verify that $P \sim (1 - \frac{t}{2}, \frac{t}{2})$.

Since the slope of $AE$ is $t-1$, the slope of $AQ$ should be $(1-t)^{-1}$. Since the coordinates of $Q$ have the form $(1+s, s(1-t)^{-1})$ for some $s$, it is straightforward to verify that

$$Q \sim \left( \frac{2 - t}{1 + (1-t)^2}, \frac{t}{1 + (1-t)^2} \right).$$
It can now be checked that the slope of each of $BQ$ and $BP$ is $t(2 - t)^{-1}$, which yields the result.

(b) The result fails if $A \sim (0, 2), B \sim (0, 0), C \sim (1, 0), D \sim (1, 2)$. If $E \sim (1, 1)$, then $P \sim (\frac{3}{5}, \frac{4}{5})$ and $Q \sim (\frac{3}{2}, \frac{1}{2})$. 