Canadian Mathematical Society

2019 Canadian Open Mathematics Challenge

A competition of the Canadian Mathematical Society and supported by the Actuarial Profession.

CMS SMC

Expertise. Insight. Solutions.

SOCIETY OF ACTUARIES

Student Identification

Please print clearly and complete all information below. Failure to print legibly or provide complete information may result in your exam being disqualified. This exam is not considered valid unless it is accompanied by your test supervisor's signed declaration.

1. Given/First Name: (Required)

2. Sur-/Last Name: (Required)

3. Are you currently registered in full-time attendance at an elementary, secondary or Cégep school, or home schooled and have been since at least September 15th of this year? (Required for qualification) ☐ **Yes** ☐ No

4. Are you a Canadian Citizen or a Permanent Resident of Canada (regardless of your current address)? (Required for qualification) ☐ **Yes** ☐ No

5. Date of Birth: (Required)

Y Y Y Y M M D D

6. Grade: (Required)

☐ 8 ☐ 10 ☐ 12
☐ 9 ☐ 11 ☐ Cégep
other: ____________

7. Gender: (Optional)

☐ Female
☐ Male

8. Email Address: (Optional)

9. Have you heard of actuary as a profession before participating in the Canadian Open Mathematics Challenge? ☐Yes ☐No

10. Have you considered pursuing a career as an actuary? ☐Yes ☐No ☐Not sure

11. Would you be interested in learning more about becoming an actuary? ☐Yes ☐No ☐Not sure

DO NOT PHOTOCOPY BLANK EXAMS!
Each page of each copy is uniquely pre-coded to facilitate computer-assisted marking.

Question A1 (4 points)

Shawn's password to unlock his phone is four digits long, made up of two 5s and two 3s. How many different possibilities are there for Shawn's password?

Your solution:

Your final answer:

Question A2 (4 points)

Triangle ABC has integer side lengths and perimeter 7. Determine all possible lengths of side AB.

Your solution:

Your final answer:

Question A3 (4 points)

Uniquely-identified page
NO PHOTOCOPIES!

If a and b are positive integers such that $a = 0.6\,b$ and $\gcd(a, b) = 7$, find $a + b$.

Your solution:

Your final answer:

Question A4 (4 points)

The equations $|x|^2 - 3|x| + 2 = 0$ and $x^4 - ax^2 + 4 = 0$ have the same roots. Determine the value of a.

Your solution:

Your final answer:

Question B1 (6 points)

John walks from home to school with a constant speed, and his sister Joan bikes twice as fast. The distance between their home and school is 3 km. If Joan leaves home 15 minutes after John then they arrive to school at the same time. What is the walking speed (in km/h) of John?

Your solution:

Your final answer:

Question B2 (6 points)

What is the largest integer n such that the quantity

$$\frac{50!}{(5!)^n}$$

is an integer?

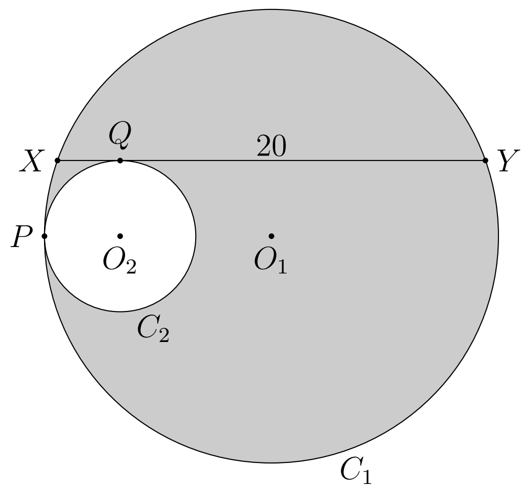
Note: Here $k! = 1 \times 2 \times 3 \times \cdots \times k$ is the product of all integers from 1 to k. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

Your solution:

Your final answer:

Question B3 (6 points)

In the diagram below circles C_1 and C_2 have centres O_1 and O_2. The radii of the circles are respectively r_1 and r_2 with $r_1 = 3r_2$. C_2 is internally tangent to C_1 at P. Chord XY of C_1 has length 20, is tangent to C_2 at Q and is parallel to O_2O_1. Determine the area of the shaded region: that is, the region inside C_1 but not C_2.



Your solution:

Your final answer:

Question B4 (6 points)

Uniquely-identified page
NO PHOTOCOPIES!

Bob and Jane hold identical decks of twelve cards, three of each colour: red, green, yellow, and blue. Bob and Jane shuffle their decks and then take turns dealing one card at a time onto a pile, with Jane going first. Find the probability that Jane deals *all* her red cards before Bob deals *any* of his red cards.
Give your answer in the form of a fraction in lowest terms.

Your solution:

Your final answer:

Question C1 (10 points)

The function f is defined on the natural numbers $1, 2, 3, \ldots$ by $f(1) = 1$ and

$$f(n) = \begin{cases} f\left(\frac{n}{10}\right) & \text{if } 10 \mid n, \\ f(n-1) + 1 & \text{otherwise.} \end{cases}$$

Note: The notation $b \mid a$ means integer number a is divisible by integer number b.

(a) Calculate $f(2019)$.

(b) Determine the maximum value of $f(n)$ for $n \leq 2019$.

(c) A new function g is defined by $g(1) = 1$ and

$$g(n) = \begin{cases} g\left(\frac{n}{3}\right) & \text{if } 3 \mid n, \\ g(n-1) + 1 & \text{otherwise.} \end{cases}$$

Determine the maximum value of $g(n)$ for $n \leq 100$.

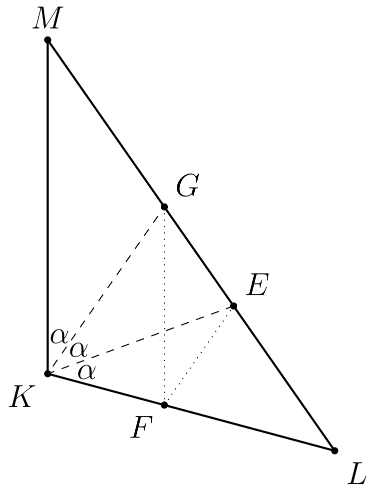
Your solution:

Question C2 (10 points)

(a) Let $ABCD$ be an isosceles trapezoid with $AB = CD = 5$, $BC = 2$, $AD = 8$. Find the height of the trapezoid and the length of its diagonals.

(b) For the trapezoid introduced in (a), find the exact value of $\cos\angle ABC$.

(c) In triangle KLM, let points G and E be on segment LM so that $\angle MKG = \angle GKE = \angle EKL = \alpha$. Let point F be on segment KL so that GF is parallel to KM. Given that $KFEG$ is an isosceles trapezoid and that $\angle KLM = 84°$, determine α.



Your solution:

Question C3 (10 points)

Let N be a positive integer. A "good division of N" is a partition of $\{1, 2, \ldots, N\}$ into two disjoint non-empty subsets S_1 and S_2 such that the sum of the numbers in S_1 equals the product of the numbers in S_2. For example, if $N = 5$, then

$$S_1 = \{3, 5\}, \qquad S_2 = \{1, 2, 4\}$$

would be a good division.

(a) Find a good division of $N = 7$.

(b) Find an N which admits two distinct good divisions.

(c) Show that if $N \geq 5$, then a good division exists.

Your solution:

Question C4 (10 points)

Three players A, B and C sit around a circle to play a game in the order $A \to B \to C \to A \to \cdots$. On their turn, if a player has an even number of coins, they pass half of them to the next player and keep the other half. If they have an odd number, they discard 1 and keep the rest. For example, if players A, B and C start with $(\underline{2}, 3, 1)$ coins, respectively, then they will have $(1, \underline{4}, 1)$ after A moves, $(1, 2, \underline{3})$ after B moves, and $(\underline{1}, 2, 2)$ after C moves, etc. (Here underline indicates the player whose turn is next to move.) We call a position $(\underline{x}, y, z)$ *stable* if it returns to the same position after every 3 moves.

(a) Show that the game starting with $(\underline{1}, 2, 2)$ (A is next to move) eventually reaches $(\underline{0}, 0, 0)$.

(b) Show that any stable position has a total of $4n$ coins for some integer n.

(c) What is the minimum number of coins that is needed to form a position that is neither stable nor eventually leading to $(\underline{0}, 0, 0)$?

Your solution:

Premier Sponsors

in association with

Partners:

ASDAN China
Dalhousie University
Dept. of Mathematics & Statistics,
(University of Saskatchewan)
Maplesoft
Memorial University
University of British Columbia
University of Calgary
University of Manitoba
University of New Brunswick
University of Prince Edward Island
University of Toronto
York University

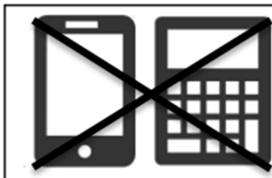
Government Sponsors:

Alberta Education
Manitoba
Nunavut
Ontario
Prince Edward Island

STUDENT INSTRUCTIONS

General Instructions:

1) Do not open the exam booklet until instructed to do so by your proctor (supervising teacher).
2) **Before the exam time starts, the proctor will give you a few minutes to fill in the Participant Identification on the cover page of the exam.** You don't need to rush. Be sure to fill in all required information fields and write legibly.
3) **Readability counts:** Make sure the pencil(s) you use are dark enough to be clearly legible throughout your exam solutions.
4) Once you have completed the exam and given it to the proctor/teacher you may leave the room.
5) The questions and solutions of the COMC exam must not be publicly discussed or shared (including online) for at least 24 hours.



Mobile phones and calculators are NOT permitted.

Exam Format:
There are three parts to the COMC to be completed in a total of 2 hours and 30 minutes:

PART A: Four introductory questions worth 4 marks each. You do not have to show your work. A correct final answer gives full marks. However, if your final answer is incorrect and you have shown your work in the space provided, you might earn partial marks.

PART B: Four more challenging questions worth 6 marks each. Marking and partial marks follow the same rule as part A.

PART C: Four long-form proof problems worth 10 marks each. Complete work must be shown. Partial marks may be awarded.

Diagrams provided are *not* drawn to scale; they are intended as aids only.

Scrap paper/extra pages: You *may* use scrap paper, but you have to throw it away when you finish your work and hand in your booklet. Only the work you do on the pages provided in the booklet will be evaluated for marking. Extra pages are not permitted to be inserted in your booklet.

Exact solutions: It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2 + \sqrt{7}$, etc., rather than as 12.566, 4.646, etc.

Awards: The names of all award winners will be published on the Canadian Mathematical Society website.