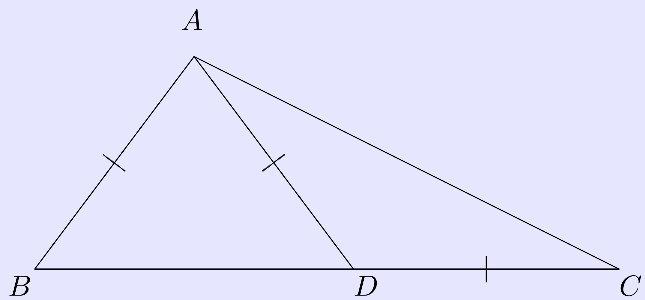
Sun Life Financial Canadian Open Mathematics Challenge 2014

Official Solutions

COMC exams from other years, with or without the solutions included, are free to download online. Please visit http://comc.math.ca/2014/practice.html

Section A – 4 marks each

1. In triangle ABC, there is a point D on side BC such that $BA = AD = DC$. Suppose $\angle BAD = 80^\circ$. Determine the size of $\angle ACB$.



Solution 1: Let $x = \angle ADB$. Then since $AB = AD$, $\angle ABD = x$. Summing the angles of $\triangle ABD$ gives $2x + 80^\circ = 180^\circ$. So $2x = 100^\circ$ and $x = 50^\circ$. Therefore, $\angle ADB = 50^\circ$. Hence, $\angle ADC = 180^\circ - \angle ADB = 180^\circ - 50^\circ = 130^\circ$.

Since $AD = DC$, $\angle ACD = \angle DAC$. Let y be this common angle. Then summing the angles of $\triangle ACD$ yield $2y + 130^\circ = 180^\circ$. Therefore, $2y = 50^\circ$. Hence, $y = 25^\circ$. Therefore, $\angle ACB = \angle ACD = 25^\circ$.

Solution 2: Since $AB = AD$, $\angle ABD = \angle ADB$. Similarly, since $DA = DC$, $\angle DAC = \angle DCA$. Let $x = \angle ABD = \angle ADB$ and $y = \angle DAC = \angle DCA$. Since $\angle DCA = \angle BCA$ we need to determine the value of y.

By external angle theorem on $\triangle ADC$, $x = 2y$. Summing the angles of $\triangle ABD$ gives the equation $4y + 80^\circ = 180^\circ$. Therefore, $4y = 100^\circ$, and $y = 25^\circ$.

2. The equations $x^2 - a = 0$ and $3x^4 - 48 = 0$ have the same real solutions. What is the value of a?

Solution 1: The left hand side of the equation $3x^4 - 48 = 0$ can be factored as $3(x^4 - 16) = 3(x^2 - 4)(x^2 + 4) = 3(x - 2)(x + 2)(x^2 + 4)$.

Thus, the real solutions of the equation $3x^4 - 48 = 0$ are $x = \pm 2$, so $x^2 = 4$, and $a = 4$.

Solution 2: The equation $3x^4 - 48 = 0$ can be factored as $3(x^2 - 4)(x^2 + 4) = 0$. Since this must have the same real solutions as $x^2 - a$ we must have $x^2 - a = x^2 - 4$ and so $a = 4$.

3. A positive integer m has the property that when multiplied by 12, the result is a four-digit number n of the form $20A2$ for some digit A. What is the 4 digit number, n?

Solution 1: For a number to be divisible by 3, the sum of the digits must be a multiple of 3, so $3|(A + 4)$, which means $A \in \{2, 5, 8\}$.

For a number to be divisible by 4, the number formed by the last two digits must be divisible by 4, so $4|(10A + 2)$. This gives $A \in \{1, 3, 5, 7, 9\}$.

The only number in common is $A = 5$, so $n = 2052$.

Solution 2: The number $n - 12$ is divisible by 5, so n is of the form $60k + 12$ for some k.

$60 \times 34 + 12 = 2052$, so $n = 2052$. It is easy to verify that no other value of k works.

4. Alana, Beatrix, Celine, and Deanna played 6 games of tennis together. In each game, the four of them split into two teams of two and one of the teams won the game. If Alana was on the winning team for 5 games, Beatrix for 2 games, and Celine for 1 game, for how many games was Deanna on the winning team?

Solution: Each game has two winners, to there will be a total of $6 \times 2 = 12$ winners.

Let A, B, C, D be the number of wins for Anna, Beatrixm Celine, and Deanna respectively. Then the total number of wins can also be expressed as $A + B + C + D$. These two quantities are equal, so $A + B + C + D = 12$.

With $A = 5, B = 2, C = 1$ we have $D = 12 - 5 - 2 - 1 = 4$. .

Section B – 6 marks each

1. The area of the circle that passes through the points $(1, 1), (1, 7)$, and $(9, 1)$ can be expressed as $k\pi$. What is the value of k?

Solution: Consider the triangle formed by the three given points. The triangle has two sides parallel to the axes, so it is right angled. The sides parallel to the axes have lengths $7 - 1 = 6$ and $9 - 1 = 8$. By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{6^2 + 8^2} = 10$.

Since the triangle formed by the three points is right angled, the hypotenuse is the diameter of the circle passing through those points. Thus, the area of the circle is $5^2\pi = 25\pi$ so $k = 25$.

2. Determine all integer values of n for which $n^2 + 6n + 24$ is a perfect square.

Solution: Suppose $n^2+6n+24 = a^2$. We can write this as $(n+3)^2+15 = a^2$. Letting $x = n+3$ we can rewrite the above equation as $a^2 - x^2 = 15$, which we can factor as $(a-x)(a+x) = 15$.

Without loss of generality we may assume that a is positive, since we get the same values of x no matter whether we use the positive or negative a value. Since a is positive, at least one of $a+x, a-x$ is also positive. Since the product of $a+x$ and $a-x$ is positive, this means that both must therefore be positive. This gives us that $(a + x, a - x) \in \{(1, 15), (3, 5), (5, 3), (15, 1)\}$.

These give solutions for (a, x) of $(8, 7), (4, 1), (4, -1), (8, -7)$. The corresponding n values for these are $4, -2, -4, -10$.

3. 5 Xs and 4 Os are arranged in the below grid such that each number is covered by either an X or an O. There are a total of 126 different ways that the Xs and Os can be placed. Of these 126 ways, how many of them contain a line of 3 Os and no line of 3 Xs?
A line of 3 in a row can be a horizontal line, a vertical line, or one of the diagonal lines $1 - 5 - 9$ or $7 - 5 - 3$.

1	2	3
4	5	6
7	8	9

Solution: If we have a horizontal (or vertical) line of all Os, then since there are 5Xs for the other two lines, there must be a horizontal (or vertical) line of all Xs. Thus, our line of 3 Os must be a diagonal.

When one of the diagonal lines is all Os, then no other line can be all Xs, since each diagonal line intersects all other lines. Thus, each configuration with one diagonal line of Os is a desired solution.

When we have the diagonal line $1-5-9$, there are 6 places that the last O could be: $2, 3, 4, 6, 7, 8$. Each of these will give a valid solution. Similarly, we have 6 solutions when we have the diagonal line $7-5-3$.

It is not possible for both diagonal lines to have only Os, since there are only 4 Os, thus we have not counted the same configuration twice. Thus 12 of the 126 ways contain a line of 3 Os and no line of 3 Xs.

4. Let $f(x) = \dfrac{1}{x^3+3x^2+2x}$. Determine the smallest positive integer n such that

$$f(1)+f(2)+f(3)+\cdots+f(n) > \frac{503}{2014}.$$

Solution 1: Note that $x^3+3x^2+2x = x(x+1)(x+2)$. We first write $1/(x^3+3x^2+2x)$ as

$$\frac{a}{x}+\frac{b}{x+1}+\frac{c}{x+2}$$

where a, b, c are real numbers.

This expression simplifies to

$$\frac{a(x+1)(x+2)+bx(x+2)+cx(x+1)}{x(x+1)(x+2)} = \frac{(a+b+c)x^2+(3a+2b+c)x+2a}{x(x+1)(x+2)}.$$

Hence, $a+b+c=0, 3a+2b+c=0$ and $2a=1$.

From the last equation, we get $a=1/2$. The first two equations simplifies to $b+c=-1/2$ and $2b+c=-3/2$. Subtracting the former from the latter yields $b=-1$. Hence, $c=1/2$. Therefore,

$$f(x) = \frac{1}{2}\left(\frac{1}{x}-\frac{2}{x+1}+\frac{1}{x+2}\right).$$

We now focus on the sum $f(1)+f(2)+f(3)+\ldots+f(n)$. This is equal to

$$\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n+2}\right).$$

All terms except $1, 1/2, 1/(n+1), 1/(n+2)$ cancel out completely. Hence, the result simplifies to

$$\frac{1}{2}-\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{(n+1)}+\frac{1}{2}\cdot\frac{1}{n+2} = \frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

We can now solve for the desired answer, i.e. find the smallest positive integer n such that

$$\frac{1}{4}-\frac{1}{2(n+1)(n+2)} > \frac{503}{2014}.$$

This simplifies to

$$\frac{1}{2(n+1)(n+2)} < \frac{1}{4} - \frac{503}{2014} = \frac{2}{4 \cdot 2014} = \frac{1}{2 \cdot 2014},$$

which is equivalent to $(n+1)(n+2) > 2014$.

By trial and error and estimation, we see that $44 \cdot 45 = 1980$ and $45 \cdot 46 = 2070$. Hence, $n = 44$ is the desired answer.

Solution 2: We obtain $(n+1)(n+2) > 2014$ as in Solution 1. This simplifies to $n^2 + 3n - 2012 > 0$. By the quadratic formula, we obtain

$$n > \frac{-3 + \sqrt{3^2 + 4 \cdot 2012}}{2} = \frac{-3 + \sqrt{8057}}{2}.$$

The largest positive integer less than $\sqrt{8057}$ is 89. Hence, $n > (-3 + 89)/2 = 43$. Therefore, $n = 44$ is the smallest positive integer satisfying the given eqation.

Section C – 10 marks each

1. A sequence of the form $\{t_1, t_2, ..., t_n\}$ is called *geometric* if $t_1 = a,\ t_2 = ar,\ t_3 = ar^2,\ \ldots\ ,\ t_n = ar^{n-1}$. For example, {1, 2, 4, 8, 16} and {1, -3, 9, -27} are both geometric sequences. In all three questions below, suppose $\{t_1, t_2, t_3, t_4, t_5\}$ is a geometric sequence of real numbers.

   (a) If $t_1 = 3$ and $t_2 = 6$, determine the value of t_5.

   (b) If $t_2 = 2$ and $t_4 = 8$, determine all possible values of t_5.

   (c) If $t_1 = 32$ and $t_5 = 2$, determine all possible values of t_4.

Solution:

(a) $t_1 = 3 = a$ and $t_2 = ar = 6$, so $r = 6/3 = 2.$.
This gives $t_5 = 3 \times 2^4 = 48$.

(b) $t_2 = 2 = ar$ and $t_4 = 8 = ar^3$, Dividing the two equations gives $r^2 = 4$, so $r = \pm 2$.
When $r = 2$ we have $a = 1$, so $t_5 = 2^4 = 16.$.
When $r = -2$ we have $a = -1$, so $t_5 = -1 \times 2^4 = -16$.

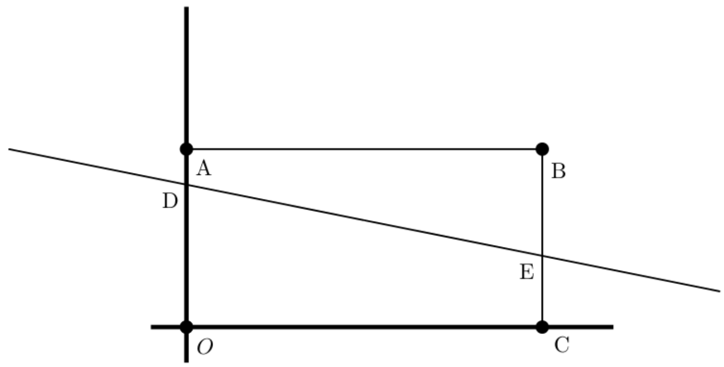
(c) We have $t_1 = 32 = a$ and $t_5 = 2 = ar^4$. This gives $a = 32$, and $r^4 = \frac{1}{16}$.
When $r^4 = \frac{1}{16}$ we get $r^2 = \frac{1}{4}$ or $r^2 = \frac{-1}{4}$.
When $r^2 = \frac{-1}{4}$, r is not a real number, so this is not a valid sequence.
When $r^2 = \frac{1}{4}$ we get $r = \pm\frac{1}{2}$.
This gives $t_4 = 32 \times \frac{1}{8} = 4$ and $t_4 = 32 \times \frac{-1}{8} = -4$.

2. The line L given by $5y+(2m-4)x-10m=0$ in the xy-plane intersects the rectangle with vertices $O(0,0)$, $A(0,6)$, $B(10,6)$, $C(10,0)$ at D on the line segment OA and E on the line segment BC.

   (a) Show that $1 \le m \le 3$.

   (b) Show that the area of quadrilateral $ADEB$ is $\frac{1}{3}$ the area of rectangle $OABC$.

   (c) Determine, in terms of m, the equation of the line parallel to L that intersects OA at F and BC at G so that the quadrilaterals $ADEB$, $DEGF$, $FGCO$ all have the same area.



Solution:

(a) Since D is on OA, the x-coordinate of D is 0. The y-coordinate D is then the solution to the equation $5y-10m=0$, i.e. $y=2m$. Hence L intersects OA at $D(0,2m)$. For D to be on OA, $0 \le 2m \le 6$, or equivalently $0 \le m \le 3$.

Similarly, the x-coordinate of E is 10, so the y-coordinate is the solution to $5y+(2m-4)(10)-10m=0$, whose solutions is $y=8-2m$. Hence $0 \le 8-2m \le 6$ or equivalently $1 \le m \le 4$.

Thus $0 \le m \le 3$ and $1 \le m \le 4$ so $1 \le m \le 3$.

(b) Observe $ADEB$ is a trapezoid with base AB and parallel sides are AD and BE, so its area is

$$\overline{AB} \cdot \frac{\overline{AD}+\overline{BE}}{2} = 10 \cdot \frac{(6-2m)+(6-(8-2m))}{2} = 10 \cdot \frac{4}{2} = 20,$$

and since the area of $OABC$ is $6 \cdot 10$, the result follows.

(c) In order for the quadrilaterals to have equal area, it is sufficient to demand $FGCO$ has area 20 (i.e. $\frac{1}{3}$ the area of $OABC$).

Let $M(5,b)$ be the midpoint of F and G. Then the average of the y-coordinates of F and G is b, so the area of $FGCO$ is $b \cdot 10 = 10b$, so $b=2$. Hence the point $M(5,2)$ is on this line.

The slope of this line is the same as L, so it is given by $\dfrac{4-2m}{5}$.

Thus the line is

$$y = \left(\frac{4-2m}{5}\right)x + (2m-2).$$

3. A local high school math club has 12 students in it. Each week, 6 of the students go on a field trip.

   (a) Jeffrey, a student in the math club, has been on a trip with each other student in the math club. Determine the minimum number of trips that Jeffrey could have gone on.

   (b) If each pair of students have been on at least one field trip together, determine the minimum number of field trips that could have happened.

Solution:

(a) There are 11 students in the club other than Jeffrey **a**nd each field trip that Jeffrey is on has 5 other students. In order for Jeffrey to go on a field trip with each other student, he must go on at least $\lceil \frac{11}{5} \rceil = \lceil 2.2 \rceil = 3$ field trips .

To see that 3 trips is sufficient, we let the other students be $\{s_1, s_2, \ldots, s_{11}\}$. On the first trip, Jeffrey can go with $\{s_1, \ldots, s_5\}$, on the second trip with $\{s_6, \ldots, s_{10}\}$ and on the third trip with s_{11} and any other 4 students. .

(b) From part (a), we know that each student must go on at least 3 trips. Since there are 12 students in total, if we count the number of trips that each student went on, we would get a minimum of $12 \times 3 = 36$. Since 6 students attend each field trip, that means there must be at least $\frac{36}{6} = 6$ trips.

We now show that it is possible to do this with exactly 6 trips. Divide the students into 4 groups of 3 students each (groups A, B, C, D). There are 6 different pairs of groups (AB, AC, AD, BC, BD, CD). Let these pairs of groups be our 6 field trips. We see that since each group goes on a trip with each other group, that each pair of students goes on a trip together. Hence, this can be done with 6 trips.

4. A polynomial $f(x)$ with real coefficients is said to be a *sum of squares* if there are polynomials $p_1(x), p_2(x), \ldots, p_n(x)$ with real coefficients for which

$$f(x) = p_1^2(x) + p_2^2(x) + \cdots + p_n^2(x).$$

For example, $2x^4 + 6x^2 - 4x + 5$ is a sum of squares because

$$2x^4 + 6x^2 - 4x + 5 = (x^2)^2 + (x^2+1)^2 + (2x-1)^2 + (\sqrt{3})^2.$$

(a) Determine all values of a for which $f(x) = x^2 + 4x + a$ is a sum of squares.

(b) Determine all values of a for which $f(x) = x^4 + 2x^3 + (a-7)x^2 + (4-2a)x + a$ is a sum of squares, and for such values of a, write $f(x)$ as a sum of squares.

(c) Suppose $f(x)$ is a sum of squares. Prove there are polynomials $u(x), v(x)$ with real coefficients such that $f(x) = u^2(x) + v^2(x)$.

Solution:

(a) If $f(x)$ is a sum of squares of polynomials, then $f(x)$ must be nonnegative for all values of x. Completing the square gives us $f(x) = (x+2)^2 + (a-4)$.

So $f(x)$ is nonnegative for all x provided that $a - 4 \geq 0$, i.e. $a \geq 4$. This is in fact sufficient for $f(x)$ to be a sum of squares, since if $a - 4 \geq 0$ then

$$f(x) = (x+2)^2 + \left(\sqrt{a-4}\right)^2.$$

Thus $f(x)$ is a sum of squares if and only if $a \geq 4$.

(b) The sum of the coefficients of $f(x)$ is 0, so $x - 1$ is a factor. Factoring this out we have

$$f(x) = (x-1)[x^3 + 3x^2 + (a-4)x - a].$$

Since the sum of the coefficients of $x^3 + 3x^2 + (a-4)x - a$ is also 0, $x - 1$ is a factor of it. Factoring this out we have

$$f(x) = (x-1)^2(x^2 + 4x + a).$$

If $f(x)$ is a sum of squares, then it must be nonnegative. Since $(x-1)^2$ is always nonnegative, we require $x^2 + 4x + a$ to be nonnegative, which as in part (a), requires $a \geq 4$.

For such a, we have

$$\begin{aligned} f(x) &= (x-1)^2(x^2+4x+a) \\ &= (x-1)^2\left((x+2)^2 + \left(\sqrt{a-4}\right)^2\right) \\ &= [(x-1)(x+2)]^2 + [\sqrt{a-4}(x-1)]^2. \end{aligned}$$

Hence $f(x)$ is a sum of squares if and only if $a \geq 4$, and we can express $f(x)$ as a sum of squares as shown above.

(c) Suppose $f(x)$ is a sum of squares. Since $f(x)$ is nonnegative for all x, its leading coefficient must be positive, and we can therefore assume it is monic by factoring out the square root of its leading coefficient. Now the non-real roots of $f(x)$ come in pairs (complex conjugates), so $f(x)$ factors into a product of linear and irreducible quadratic polynomials over the reals, raised to certain powers, say

$$f(x) = \prod_{i=1}^{m} p_i(x)^{k_i} \prod_{j=1}^{n} q_i(x)^{j_i},$$

where p_i's are the distinct linear polynomials, q_i's are the distinct irreducible quadratic polynomials, and $k_i, j_i > 1$ for each i.

For each i, let $q_i(x) = x^2 + a_i x + b_i$. Since $q_i(x)$ is irreducible over the reals, $a_i^2 - 4b_i < 0$, so $q_i(x)$ can be written as a sum of squares, namely

$$q_i(x) = \left(x + \frac{a_i}{2}\right)^2 + \left(\sqrt{b_i - \frac{a_i^2}{4}}\right)^2.$$

Now for each i, let $p_i(x) = x - c_i$.

We claim k_i is even for all i. Suppose otherwise, and say the exponents $k_1, k_2, \dots, k_m$ which are odd are $k_{i_1}, k_{i_2}, \dots, k_{i_l}$ where without loss of generality $c_{i_1} < c_{i_2} < \cdots < c_{i_l}$. Then

$$f(x) = (x - c_{i_1})(x - c_{i_2}) \cdots (x - c_{i_l}) g(x),$$

where $g(x)$ is nonnegative for all x (since $g(x)$ is the product of irreducible quadratics which we proved are sums of squares and hence nonnegative, and even powers of linear polynomials). Then $f\left(\frac{c_{i_l} + c_{i_{l-1}}}{2}\right) < 0$, which is impossible if $f(x)$ is a sum of squares.

We then deduce that

$$f(x) = h(x)^2 \prod_{j=1}^{n} \left(r_j^2(x) + s_j^2(x)\right), \tag{1}$$

where $h(x)$ and $r_j(x), s_j(x)$ are polynomials, for all j (namely $h(x) = \prod_{i=1}^{m} p_i(x)^{\frac{k_i}{2}}$, $r_j(x) = x + \frac{a_j}{2}$, and $s_j(x) = \sqrt{b_j - \frac{a_j^2}{4}}$.)

Now observe that for any i, j,

$$(r_i^2 + s_i^2)(r_j^2 + s_j^2) = (r_i r_j + s_i s_j)^2 + (r_i s_j - r_j s_i)^2.$$

Repeatedly applying this identity to (5) gives us polynomials $P(x), Q(x)$ for which $P^2(x) + Q^2(x) = \prod_{j=1}^{n} \left(r_j^2(x) + s_j^2(x)\right)$, and hence

$$f(x) = (h(x)P(x))^2 + (h(x)Q(x))^2.$$

