A1 Determine the positive integer n that satisfies the following equation:

$$\frac{1}{2^{10}} + \frac{1}{2^9} + \frac{1}{2^8} = \frac{n}{2^{10}}.$$

Solution

Adding the left hand side of the given equation with with a common denominator of 2^{10}, we have,

$$\frac{1}{2^{10}} + \frac{1}{2^9} + \frac{1}{2^8} = \frac{1}{2^{10}} + \frac{2}{2^{10}} + \frac{2^2}{2^{10}} = \frac{1+2+4}{2^{10}} = \frac{7}{2^{10}}.$$

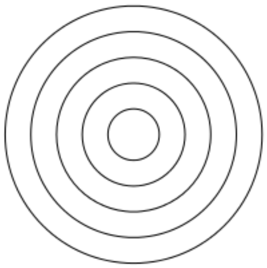
Therefore, $n = 7$.

A2 Determine the *positive* integer k for which the parabola $y = x^2 - 6$ passes through the point (k, k).

Solution

If the curve passes through the point (k, k), then we can substitute $x = k, y = k$ into the given equation to get $k^2 - k - 6 = 0$. We can factor this as $(k - 3)(k + 2) = 0$, so $k = 3$ or $k = -2$. Since we want the positive value of k, we get $k = 3$.

A3 In the figure below, the circles have radii 1, 2, 3, 4, and 5. The total area that is contained inside an *odd* number of these circles is $m\pi$ for a positive number m. What is the value of m?



Solution

A point is inside an odd number of circles if it is in the outermost ring, the third ring, or the middle circle. The area of the middle circle is π. The third ring is the area contained in the circle of radius 3 but not contained in the circle of radius 2. The area of the third ring is $3^2\pi - 2^2\pi = 5\pi$. The outer ring is the area contained in the circle of radius 5 but not contained in the circle of radius 4. The area of the fifth ring is $5^2\pi - 4^2\pi = 9\pi$. Thus, the total area is $\pi + 5\pi + 9\pi = 15\pi$, so $m = 15$.

A4 A positive integer is said to be bi-digital if it uses two different digits, with each digit used exactly twice. For example, 1331 is bi-digital, whereas 1113, 1111, 1333, and 303 are not. Determine the exact value of the integer b, the number of bi-digital positive integers.

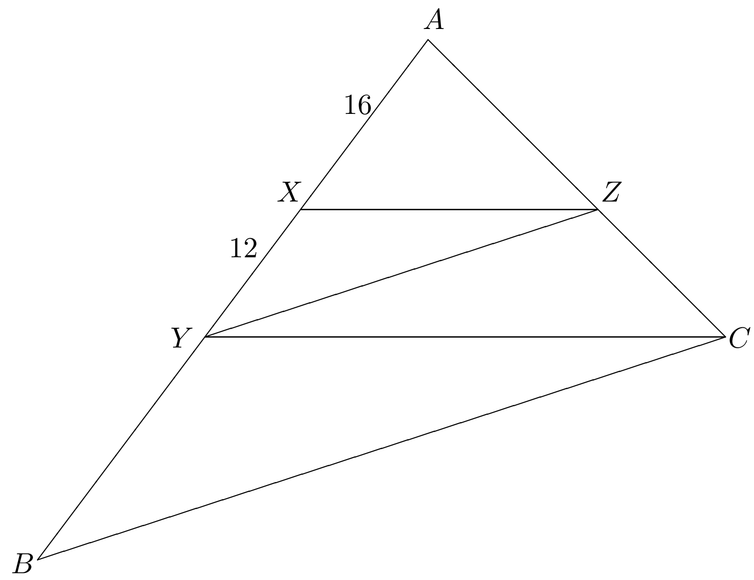
Solution 1

There are 9 choices for what the left-most digit of the number is (it cannot be 0) and there are 3 choices for where the second copy of this digit is. There are 9 possibilities for the digit that fills the remaining positions. Thus, $b = 9 \times 3 \times 9 = 243$.

Solution 2

We consider two cases. Either 0 is one of the digits, or it is not. If 0 is not one of the digits, then we have $\binom{9}{2} = 36$ ways to choose 2 digits which are not 0. There are $\frac{4!}{(2!)^2} = 6$ ways to arrange these digits, for a total of 216 numbers. If 0 is one of the digits, it cannot be the first digit of the number, since then the number would have fewer than 4 digits. In this case, there are $\binom{9}{1} = 9$ ways to choose the other digit. The first digit must be the non-zero digit and there are 3 places for the other non-zero digit, so there are 27 such numbers. Thus, $b = 216 + 27 = 243$.

B1 Given a triangle ABC, X, Y are points on side AB, with X closer to A than Y, and Z is a point on side AC such that XZ is parallel to YC and YZ is parallel to BC. Suppose $AX = 16$ and $XY = 12$. Determine the length of YB.



Solution

Triangles AXZ and AYC are similar, so $AZ : AX = ZC : XY$ and so $AZ/ZC = 4/3$. Also, triangles AYZ and ABC are similar, so $AZ : ZC = 28 : YB$. Combining the two results gives $4/3 = \frac{28}{YB}$ so $YB = 21$.

B2 There is a unique triplet of positive integers (a, b, c) such that $a \le b \le c$ and

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}.$$

Determine $a + b + c$.

Solution:

Note that $\frac{1}{4} < \frac{25}{84} < \frac{1}{3}$. Therefore, $a \ge 4$. But if $a \ge 5$, then $b, c \ge 5$. Consequently,

$$\frac{25}{84} = \frac{1}{a} + \frac{1}{ab} + \frac{1}{abc} \le \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} = \frac{5^2+5+1}{5^3} = \frac{31}{125} < \frac{1}{4} < \frac{25}{84},$$

which is a contradiction. Therefore, $a \ngeq 5$. Hence, $a = 4$.

Substituting this into the equation given in the problem yields

$$\frac{25}{84} = \frac{1}{4} + \frac{1}{4b} + \frac{1}{4bc}.$$

Multiplying both sides by 4 and rearranging yields

$$\frac{4}{21} = \frac{1}{b} + \frac{1}{bc}. \tag{1}$$

Note that $\frac{1}{6} < \frac{4}{21} < \frac{1}{5}$. Therefore, $b \ge 6$. If $b \ge 7$, then $c \ge 7$. Hence,

$$\frac{4}{21} = \frac{1}{b} + \frac{1}{bc} \le \frac{1}{7} + \frac{1}{7^2} = \frac{7+1}{7^2} = \frac{8}{49} < \frac{1}{6} < \frac{4}{21},$$

which is a contradiction. Therefore, $b \ngeq 7$. Consequently, $b = 6$. Substituting this into (1) yields

$$\frac{4}{21} = \frac{1}{6} + \frac{1}{6c}.$$

Multiplying both sides by 6 and rearranging yields

$$\frac{1}{7} = \frac{1}{c}.$$

Therefore, $c = 7$.

Hence, $(a, b, c) = (4, 6, 7)$, which yields $a + b + c = 17$.

B3 Teams A and B are playing soccer until someone scores 29 goals. Throughout the game the score is shown on a board displaying two numbers – the number of goals scored by A and the number of goals scored by B. A mathematical soccer fan noticed that several times throughout the game, the sum of all the digits displayed on the board was 10. (For example, a score of 12 : 7 is one such possible occasion). What is the maximum number of times throughout the game that this could happen?

Solution 1

When the sum of all the digits on the scoreboard is 10, the sum of the scores must be 1 more than a multiple of 9. The highest possible sum of the scores is $29 + 28 = 57$. The numbers less than 57 that are 1 more than a multiple of 9 are 1, 10, 19, 28, 37, 46, and 55. If the sum of the scores is 1, then the sum of the digits is 1, not 10. If the sum of the scores is 55, then the scores are 26 and 29 or 27 and 28, both of which have a digit sum of 19. Thus, we cannot have this happen more than 5 times.

We see that the scores $(5, 5), (5, 14), (14, 14), (23, 14), (23, 23)$ each have a digit sum of 10, and can all be acheived in the same game. Thus, the maximum number of times is 5.

Solution 2

Denote by (a_1a_2, b_1b_2) the score displayed on the board where a_1, a_2, b_1, b_2 are digits (we allow a_1 and b_1 to be 0), and a_1a_2, b_1b_2 are the numbers of goals scored by the two teams. We will call a score *good* if $a_1 + a_2 + b_1 + b_2 = 10$.

Lemma: Suppose scores (x, y) and (z, t) occurred throughout the game. Then at most one of $x > z$ and $y < t$ can hold.

Proof: Suppose that $x > z$. Then the first team scored x goals after it scored z goals, so the score (x, y) occurred later in the game than the score (z, t). Therefore $y \geq t$, and the result follows.

We now show that we cannot have two good scores occurring throughout the game of the form (a_1a_2, a_1b_2) and (a_1a_2', a_1b_2'). Suppose the scores did occur; then $a_2 + b_2 = a_2' + b_2'$. WLOG $a_2 > a_2'$. Then $b_2 < b_2'$; hence $a_1a_2 > a_1a_2'$; $a_1b_2 < a_1b_2'$, which is impossible by the Lemma.

We next claim that if $a_1 > b_1$, then at most one of the good scores (a_1a_2, b_1b_2), (b_1a_2', a_1b_2') could occur throughout the game. This follows immediately from the Lemma since $a_1a_2 > b_1a_2'$; $a_1b_2' > b_1b_2$.

Since the game ends when someone scores 29 goals, the tens digit for each team is 0, 1, or 2. By the first claim have at most nine possibilities for the good scores: $(0a, 0b)$, $(0a, 1b)$, $(0a, 2b)$, $(1a, 0b)$, $(1a, 1b)$, $(1a, 2b)$, $(2a, 0b)$, $(2a, 1b)$, $(2a, 2b)$ for some digits a, b (possibly different for each case). By the second claim, at most one of $(0a, 1b)$ and $(1a, 0b)$; $(0a, 2b)$ and $(2a, 0b)$; $(1a, 2b)$ and $(2a, 1b)$ can occur, eliminating three possibilities. Furthermore, if $(0a, 2b)$ or $(2a, 0b)$ occurred then $(1a, 1b)$ could not occur and vice versa (since if WLOG $(0a, 2b)$ occurred, then the second team had at least 20 points by the time the first team got to 10 points). This eliminates one more possibility.

Hence at most 9 - 3 - 1 = 5 good scores occurred. It remains to give an example when this occurrence is indeed possible. One such example is $(3, 7), (8, 11), (14, 14)$, $(16, 21), (23, 23)$.

B4 Let a be the largest real value of x for which $x^3 - 8x^2 - 2x + 3 = 0$. Determine the integer closest to a^2.

Solution 1

Since the equation has degree 3, there are at most 3 values of x for which it will hold.

Let $f(x) = x^3 - 8x^2 - x + 3$, and b, c the other two roots of $f(x)$.

Note that

$$f(-1) = (-1)^3 - 8(-1)^2 - 2(-1) + 3 = -4 < 0$$

and

$$f\left(\frac{-1}{2}\right) = \left(\frac{-1}{2}\right)^3 - 8\left(\frac{-1}{2}\right)^2 - 2 \cdot \frac{-1}{2} + 3 = \frac{-1}{8} - 2 + 1 + 3 > 0.$$

Hence, there is a root between -1 and $-1/2$.

Similarly,

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 8\left(\frac{1}{2}\right)^2 - 2 \cdot \frac{1}{2} + 3 = \frac{1}{8} - 2 - 1 + 3 = \frac{1}{8} > 0$$

and

$$f(1) = 1 - 8 - 2 + 3 = -6 < 0.$$

Hence, there is a root between $1/2$ and 1.

Hence, suppose $-1 < b < -1/2$ and $1/2 < c < 1$.

Consider the quantity $a^2+b^2+c^2$. By the factor theorem, $x^3-8x^2-2x+3 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$. Therefore, $a+b+c = 8$ and $ab+bc+ca = -2$. Then $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 8^2 - 2 \cdot (-2) = 68$.

Now, we consider the quantity $b^2 + c^2$. Since $b < -1/2$ and $c > 1/2$, $b^2 + c^2 > 1/2$. Now we need an upper bound on $b^2 + c^2$. Note that

$$f\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^3 - 8\left(\frac{1}{\sqrt{2}}\right)^2 - 2\left(\frac{1}{\sqrt{2}}\right) + 3 = \frac{1}{2\sqrt{2}} - 4 - \frac{2}{\sqrt{2}} + 3 = \frac{-3}{2\sqrt{2}} - 1 < 0.$$

Since $f(1/2) > 0$, $1/2 < c < 1/\sqrt{2}$.

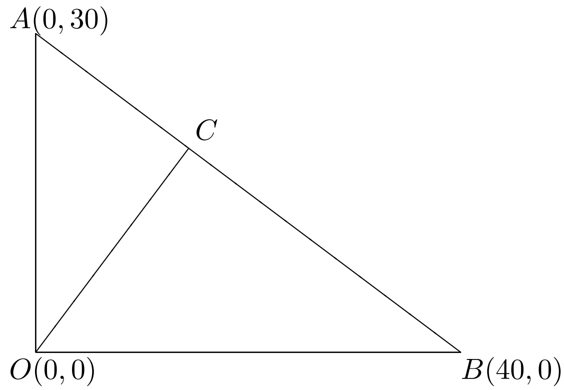
Therefore, $b^2 + c^2 < 1 + 1/2 = 3/2$. Since $1/2 < b^2 + c^2 < 3/2$ and $a^2 + b^2 + c^2 = 68$, $66.5 < a^2 < 67.5$. Therefore, the integer closest to a^2 is 67.

Solution 2

As in solution 1, we can verify that there are two values of x between -1 and 1 for which the equation holds. Note that since the equation is cubic there are at most 3 distinct solutions.

We can rewrite the equation as $x^2(x - 8) = 2x - 3$, which simplifies to $x^2 = 2 + \frac{13}{x-8}$. Letting $x = 8.2$ we get the left hand side is $8.2^2 = 67.24$ and the right side is $2 + \frac{13}{.2} = 67$. As we decrease x, from 8.2 to 8.1, the left hand side decreases from 67.24 to 65.61 and the right hand side increases from 67 to 132. Since both functions are continuous, there is a point between where they will have the same value, and that value will be between 67 and 67.24. Thus, the integer closest to x^2 is 67.

C1 In the diagram, $\triangle AOB$ is a triangle with coordinates $O = (0, 0)$, $A = (0, 30)$, and $B = (40, 0)$. Let C be the point on AB for which OC is perpendicular to AB.



(a) Determine the length of OC.

(b) Determine the coordinates of point C.

(c) Let M be the centre of the circle passing through O, A, and B. Determine the length of CM.

Solution 1

(a) By the Pythagorean Theorem, the length of AB is $\sqrt{30^2 + 40^2} = 50$. By calculating the area of the triangle as $AB \times CO/2$ and $AO \times OB/2$ we get that $50 \times OC = 1200$, and $OC = 24$.

(b) Since OC is perpendicular to AB, angle ACO is a right angle. Thus, triangle ACO is similar to triangle AOB, so $AC : AO = AO : AB$ and $AC = 18$. So point C is $\frac{18}{50}$ of the way along the line from A to B. Thus, the coordinates are $\left(\frac{18}{50} \times 40, \frac{32}{50} \times 30\right) = \left(\frac{72}{5}, \frac{96}{5}\right)$

(c) Since the angle AOB is a right angle, AB is a diameter of the circle through O, A, and B. Thus, M must be the midpoint of the line AB. We already calculated that $AC = 18$, and we know that $AM = AB/2 = 25$, so $CM = AM - AC = 25 - 18 = 7$.

Solution 2

(a) By the Pythagorean Theorem, the length of AB is $\sqrt{30^2 + 40^2} = 50$. Since OC is perpendicular to AB, angle ACO is a right angle and thus $\frac{AO}{OC} = \frac{AB}{OB}$, so $OC = \frac{AO \times OB}{AB} = \frac{1200}{50} = 24$.

(b) The equation of the line through A and B has the form $\frac{x}{y-30} = \frac{40-0}{0-30}$, which we can rewrite as $y = -\frac{3}{4}x + 30$. The equation of the line through O and C is perpendicular to $y = -\frac{3}{4}x + 30$, so it has slope $\frac{4}{3}$ and the equation is $y = \frac{4}{3}x$. These lines intersect at the point $\left(\frac{72}{5}, \frac{96}{5}\right)$, which are the coordinates of C.

(c) Let (x, y) be the coordinates of M. Since M is the centre of a circle containing the points A, B, O we have $MA = MMO = MB$. This gives $x^2 + (y - 30)^2 = x^2 + y^2 = (x-40)^2+y^2$. The first equality gives $y = 15$ and the second equality gives $x = 20$, so $M = (20, 15)$. By the Pythagorean theorem, the length of MC is $\sqrt{\left(20 - \frac{72}{5}\right)^2 + \left(15 - \frac{96}{5}\right)^2} = \frac{\sqrt{28^2+(-21)^2}}{5} = \frac{35}{5} = 7$.

C2 (a) Determine all real solutions to $a^2 + 10 = a + 10^2$.

(b) Determine two positive real numbers $a, b > 0$ such that $a \neq b$ and $a^2 + b = b^2 + a$.

(c) Find all triples of real numbers (a, b, c) such that $a^2 + b^2 + c = b^2 + c^2 + a = c^2 + a^2 + b$.

Solution

We can rearrange the equation as follows:

$$\begin{aligned}
&a^2 - b^2 = a - b \\
&(a - b)(a + b) = (a - b) \\
&(a - b)(a + b - 1) = 0
\end{aligned}$$

This tells us that our two solutions are $a = b$ and $a = 1 - b$.

(a) By the above result, the solutions are $a = 10, a = -9$.

(b) By the above result, the pair $a = \frac{1}{4}$ and $b = \frac{3}{4}$ is such a pair of positive real numbers. Any pair of positive real numbers a, b with $a + b = 1$ will suffice.

(c) Applying the above result to the first two parts of the equality gives $a = c$ or $a = 1 - c$. Applying it to the first and third gives $b = c$ or $b = 1 - c$. Applying to the second and third gives $a = b$ or $b = 1 - a$.

Fix any real number a. Then $b = a$ or $b = 1 - a$ and $c = a$ or $c = 1 - a$. Note any pair (b, c) formed satisfies $b = c$ or $b = 1 - c$. Hence, all four solutions $(a, a, a), (a, a, 1 - a), (a, 1 - a, a), (a, 1 - a, 1 - a)$ are solutions to the given equation.

C3 Alphonse and Beryl play the following game. Two positive integers m and n are written on the board. On each turn, a player selects one of the numbers on the board, erases it, and writes in its place any positive divisor of this number as long as it is different from any of the numbers previously written on the board. For example, if 10 and 17 are written on the board, a player can erase 10 and write 2 in its place (as long as 2 has not appeared on the board before). The player who cannot make a move loses. Alphonse goes first.

(a) Suppose $m = 2^{40}$ and $n = 3^{51}$. Determine which player is always able to win the game and explain the winning strategy.

(b) Suppose $m = 2^{40}$ and $n = 2^{51}$. Determine which player is always able to win the game and explain the winning strategy.

Solution

(a) Notice that for (a) m and n have greatest common divisor equal to 1, therefore on each turn a player can always make a move of replacing the number k with its divisor l strictly less than k, as long as $l > 1$, or as long as $l = 1$ and 1 has not yet appeared on the board.

Instead of dealing with the actual numbers we will deal with the number of prime factors they have. Then, the game becomes equivalent to the following. Two numbers m and n are written on the board. On each turn a player can select a number k greater than 0 and replace it with any positive integer less than k, or replace it with 0, as long as 0 is not already written on the board. A player who cannot make a move loses.

It immediately follows that $m = 0, n = 1$ is a losing position. Therefore, $m = 0, n \geq 2$ is a winning position (since a player replaces n with 1 and wins). Furthermore, $m = 1, n \geq 1$ is a winning position (since a player replaces n with 0 and wins). Hence $m = 2, n = 2$ is a losing position; $m = 2, n \geq 3$ is a winning position; $m = 3, n = 3$ is a losing position, $m = 3, n \geq 4$ is a winning position. By induction it follows that for $k \geq 2$, $m = k, n = k$ is a losing position, while $m = k, n \geq k + 1$ is a winning position.

We are in the case of $m = 40, n = 51 \geq 41$ in the "transformed" game, thus this is a winning position and Alphonse wins.

(b) This case is different, since now m and n have more than one divisor in common. We will deal with the original game and not make any transformations. Note that m and n are both powers of 2, so throughout the whole game only powers of 2 can appear on the board.

We first note that the player who first writes down a number less than or equal to 2 loses. This is because if they write down 1, then 2 has not yet been written; the opponent on the next turn replaces the other number with 2 wins. (Note that this move is legal since at the start $m > 2, n > 2$ so at the time that 1 is written, the other number on the board must be greater than 2). If they write down 2, then 1 has not yet been written; the opponent on the next turn replaces the other number with 1 and wins.

Similarly, the player who first writes down a number less than or equal to 8 loses. This is because if they write down 4, the other player writes 8 – thus forcing the original player to write down a number less than or equal to 2 (note they cannot replace 8 with 4 since 4 has already appeared on the board). Similarly, if they write down 8, the other player writes down 4 and wins.

By induction it follows that if $m, n > 2^{2k-1}$ then the player who first writes down a number less than or equal to 2^{2k-1} loses for every positive integer k. Thus for the case $m = 2^{40}, n = 2^{53}$, the player to first write down a number less than or equal to 2^{39} loses. Therefore on his first turn, Alphonse replaces 2^{53} with 2^{41} and wins – because on her turn, Beryl is faced with 2^{40} and 2^{41} on the board and has to write down a number less than or equal to 2^{39}.

C4 For each real number x, let $[x]$ be the largest integer less than or equal to x. For example, $[5] = 5$, $[7.9] = 7$ and $[-2.4] = -3$. An *arithmetic progression* of length k is a sequence $a_1, a_2, \ldots, a_k$ with the property that there exists a real number b such that $a_{i+1} - a_i = b$ for each $1 \le i \le k-1$.

Let $\alpha > 2$ be a given irrational number. Then $S = \{[n \cdot \alpha] : n \in \mathbb{Z}\}$, is the set of all integers that are equal to $[n \cdot \alpha]$ for some integer n.

(a) Prove that for any integer $m \ge 3$, there exist m distinct numbers contained in S which form an arithmetic progression of length m.

(b) Prove that there exist no infinite arithmetic progressions contained in S.

Solution

(a) We first prove the following statement: For each positive integer m there exist positive integers $n \le m$ and x_m such that $|n\alpha - x_m| < \frac{1}{m}$.

We consider the *fractional parts* of the numbers $n\alpha$ for $n = 0, \ldots, m$, i.e., consider $\{n\alpha\} := n\alpha - [n\alpha]$. By the definition of the integer part of a real number we conclude that each $\{n\alpha\} \in [0, 1)$.

Using the pigeonhole principle we conclude that there must exist two distinct integers $0 \le n_1 < n_2 \le m$ such that both the corresponding fractional parts $\{n_1\alpha\}$ and $\{n_2\alpha\}$ belong to the *same* interval $[(k-1)/m, k/m)$, for some $k = 1, \ldots, m$. Hence $|\{n_2\alpha\} - \{n_1\alpha\}| < \frac{1}{m}$.

Thus $|n_2\alpha - [n_2\alpha] - n_1\alpha + [n_1\alpha]| < \frac{1}{m}$, and therefore letting $n := n_2 - n_1$ and $x_m := [n_2\alpha] - [n_1\alpha]$, we conclude that $|n\alpha - x_m| < \frac{1}{m}$.

Furthermore, since $0 \le n_1 < n_2 \le m$, we get that $n \le m$ is a positive integer. Also, using that $\alpha > 2$ while $n_2 > n_1$ we conclude that $x_m = [n_2\alpha] - [n_1\alpha] \ge [\alpha] \ge 2$ is also a positive integer.

As proved above, for each integer $m \ge 3$, there exist positive integers $n \le m$ and x_m such that $|n\alpha - x_m| < \frac{1}{m}$. At the expense of replacing n by $-n$ and replacing x_m by $-x_m$, we may assume that $0 < \{n\alpha\} < \frac{1}{m}$, and thus $0 < n\alpha - x_m < 1/m$.

Then $x_m = [n\alpha]$ and so, $n\alpha = x_m + \{n\alpha\}$. We deduce that for each $k \in \{1, 2, \cdots, m\}$ we have $kx_m < nk\alpha = kx_m + k\{n\alpha\} < kx_m + 1$. So, $[nk\alpha] = kx_m$, which proves that indeed the numbers $[n\alpha]$, $[2n\alpha], \cdots, [mn\alpha]$ form an arithmetic progression.

(b) Assume there exists an infinite arithmetic progression in S: $[n_1\alpha]$, $[n_2\alpha], \cdots, [n_i\alpha], \cdots$. For each $i \in \mathbb{N}$, using the fact that $[n_i\alpha] + [n_{i+2}\alpha] = 2[n_{i+1}\alpha]$, we conclude that $(n_{i+2} - 2n_{i+1} + n_i) \cdot \alpha = \{n_{i+2}\alpha\} - 2\{n_{i+1}\alpha\} + \{n_i\alpha\} \in (-2, 2)$, where in the last inequality we used the fact that the fractional part of any real number is in the interval $[0, 1)$.

However, since each $n_i \in \mathbb{Z}$ and moreover, $\alpha > 2$ we conclude that for each $i \in \mathbb{N}$ we have $n_{i+2} - 2n_{i+1} + n_i = 0$. So, $n_1\alpha, n_2\alpha, \cdots, n_i\alpha, \cdots$ is itself an arithmetic progression. Therefore, the difference of the two arithmetic progressions is another infinite arithmetic progression: $\{n_1\alpha\}$, $\{n_2\alpha\}, \cdots, \{n_i\alpha\}, \cdots$.

However, the arithmetic progressions cannot be bounded, unless their ratio is 0. Hence $\{n_2\alpha\} = \{n_1\alpha\}$, which yields that $n_2\alpha - n_1\alpha = [n_2\alpha] - [n_1\alpha] \in \mathbb{Z}$ and therefore $\alpha \in \mathbb{Q}$, which is a contradiction with our assumption (also note that $n_2 \neq n_1$ since they belong to an infinite arithmetic progression).

