The Canadian Mathematical Society La Société mathématique du Canada

The Canadian Mathematical Society

in collaboration with

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

presents the

Canadian Open Mathematics Challenge

Wednesday, November 23, 2005

Time: $2\frac{1}{2}$ hours

Calculators are NOT permitted.

Do not open this booklet until instructed to do so.
There are two parts to this paper.

PART A
This part of the paper consists of 8 questions, each worth 5 marks. You can earn full value for each question by entering the correct answer in the space provided. If you do not have the correct answer, any work you do in obtaining an answer will be considered for part marks, **provided that it is done in the space allocated** to that question in your answer booklet.

PART B
This part of the paper consists of 4 questions, each worth 10 marks. Finished solutions must be written in the appropriate location in the answer booklet. Rough work should be done separately. If you require extra pages for your finished solutions, paper will be provided by your supervising teacher. Any extra papers should be placed inside your answer booklet. Be sure to write your name and school name on any inserted pages.
Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the contest, insert the information sheet inside the answer booklet.

Canadian Open Mathematics Challenge

NOTE:	1.	Please read the instructions on the front cover of this booklet.
	2.	Write solutions in the answer booklet provided.
	3.	It is expected that all calculations and answers will be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc.
	4.	Calculators are **not** allowed.

PART A

1. Determine the value of $10^2 - 9^2 + 8^2 - 7^2 + 6^2 - 5^2 + 4^2 - 3^2 + 2^2 - 1^2$.

2. A bug in the xy-plane starts at the point $(1, 9)$. It moves first to the point $(2, 10)$ and then to the point $(3, 11)$, and so on. It continues to move in this way until it reaches a point whose y-coordinate is twice its x-coordinate. What are the coordinates of this point?

3. If $ax^3 + bx^2 + cx + d = (x^2 + x - 2)(x - 4) - (x + 2)(x^2 - 5x + 4)$ for all values of x, what is the value of $a + b + c + d$?

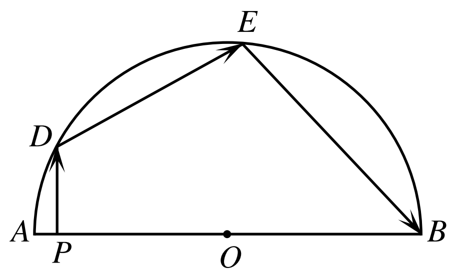
4. A fraction $\frac{p}{q}$ is in lowest terms if p and q have no common factor larger than 1.
   How many of the 71 fractions $\frac{1}{72}, \frac{2}{72}, \ldots, \frac{70}{72}, \frac{71}{72}$ are in lowest terms?

5. An office building has 50 storeys, 25 of which are painted black and the other 25 of which are painted gold. If the number of gold storeys in the top half of the building is added to the number of black storeys in the bottom half of the building, the sum is 28. How many gold storeys are there in the top half of the building?

6. In the grid shown, each row has a value assigned to it and each column has a value assigned to it. The number in each cell is the sum of its row and column values. For example, the "8" is the sum of the value assigned to the 3rd row and the value assigned to the 4th column. Determine the values of x and y.

3	0	5	6	-2
-2	-5	0	1	y
5	2	x	8	0
0	-3	2	3	-5
-4	-7	-2	-1	-9

7. In the diagram, the semi-circle has centre O and diameter AB. A ray of light leaves point P in a direction perpendicular to AB. It bounces off the semi-circle at point D in such a way that $\angle PDO = \angle EDO$. (In other words, the angle of incidence equals the angle of reflection at D.) The ray DE then bounces off the circle in a similar way at E before finally hitting the semicircle again at B. Determine $\angle DOP$.



8. The number 18 *is not* the sum of any 2 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 2 different ways, since $5+6+7=18$ and $3+4+5+6=18$. Determine a positive integer less than 400 that *is not* the sum of any 11 consecutive positive integers, but *is* the sum of consecutive positive integers in at least 11 different ways.

PART B

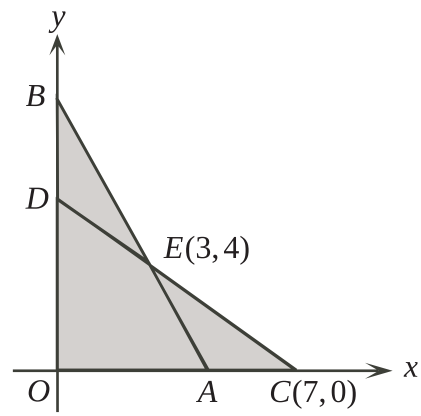
1. A line with slope -3 intersects the positive x-axis at A and the positive y-axis at B. A second line intersects the x-axis at $C(7,0)$ and the y-axis at D. The lines intersect at $E(3,4)$.

   (a) Find the slope of the line through C and E.

   (b) Find the equation of the line through C and E, and the coordinates of the point D.

   (c) Find the equation of the line through A and B, and the coordinates of the point B.

   (d) Determine the area of the shaded region.



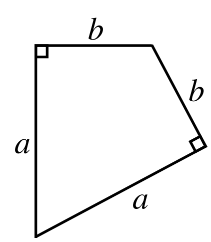
2. (a) Determine all possible ordered pairs (a,b) such that

$$\begin{aligned} a-b &= 1 \\ 2a^2+ab-3b^2 &= 22 \end{aligned}$$

   (b) Determine all possible ordered triples (x,y,z) such that

$$\begin{aligned} x^2-yz+xy+zx &= 82 \\ y^2-zx+xy+yz &= -18 \\ z^2-xy+zx+yz &= 18 \end{aligned}$$

3. Four tiles identical to the one shown, with $a>b>0$, are arranged without overlap to form a square with a square hole in the middle.

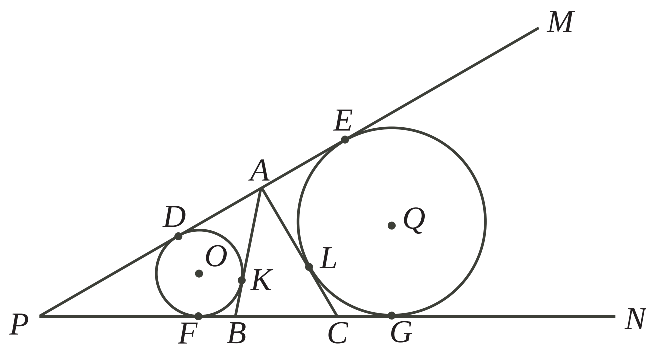


   (a) If the outer square has area $(a+b)^2$, show that the area of the inner square is $(a-b)^2$.

   (b) Determine the smallest integer value of N for which there are prime numbers a and b such that the ratio of the area of the inner square to the area of the outer square is $1:N$.

   (c) Determine, with justification, all positive integers N for which there are odd integers $a>b>0$ such that the ratio of the area of the inner square to the area of the outer square is $1:N$.

4. Triangle ABC has its base on line segment PN and vertex A on line PM. Circles with centres O and Q, having radii r_1 and r_2, respectively, are tangent to the triangle ABC externally and to each of PM and PN.



(a) Prove that the line through K and L cuts the perimeter of triangle ABC into two equal pieces.

(b) Let T be the point of contact of BC with the circle inscribed in triangle ABC. Prove that $(TC)(r_1) + (TB)(r_2)$ is equal to the area of triangle ABC.