The Canadian Mathematical Society
in collaboration with
The Center for Education
in Mathematics and Computing

The Second Canadian Open Mathematics Challenge

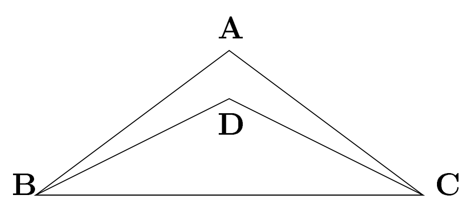
Wednesday, November 26, 1997

Solutions

Part A

Note: All questions in part A were graded out of 5 points.

1. In triangle ABC, $\angle A$ equals 120 degrees. A point D is inside the triangle such that $\angle DBC = 2\cdot\angle ABD$ and $\angle DCB = 2 \cdot \angle ACD$. Determine the measure, in degrees, of $\angle BDC$.



By letting $\angle DBC = 2x$ and $\angle DCB = 2y$, one obtains an equation involving $3x + 3y + 120$, leading to $\angle BDC = 140°$.

The average score was 4.0.

2. Solve the following system of equations:

$$xy^2 = 10^8, \ \frac{x^3}{y} = 10^{10}.$$

There are many approaches. Probably the most straightforward is to determine x in terms of y in the first equation, and then to substitute this in the second equation. The answer is $x = 10^4, \ y = 10^2$.

The average score was 3.8

3. Determine all points on the straight line which joins $(-4, 11)$ to $(16, -1)$ and whose coordinates are positive integers.

By using the given points, the slope of the line segment is $-\frac{3}{5}$. Using this slope, the points are easily determined to be $(11, 2), (6, 5)$, and $(1, 8)$.

The average score was 3.7.

4. Given three distinct digits a, b and c, it is possible, by choosing two digits at a time, to form six two-digit numbers. Determine all possible sets $\{a, b, c\}$ for which the sum of the six two-digits numbers is 484.

The six possible numbers are $10a + b, 10a + c, 10b + a, 10b + c, 10c + a, 10c + b$. Their sum is $22(a + b + c)$. From this, the acceptable sets are $\{6, 7, 9\}$ and $\{5, 8, 9\}$, since the digits are distinct.

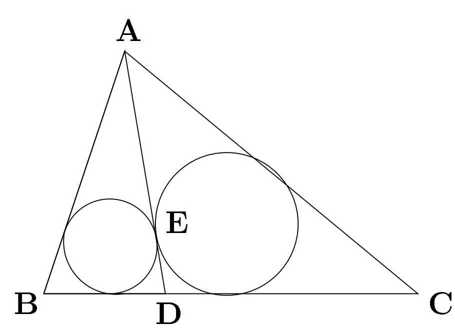
The average score was 2.8.

5. Two cubes have their faces painted either red or blue. The first cube has five red faces and one blue face. When the two cubes are rolled simultaneously, the probability that the two top faces show the same colour is $\frac{1}{2}$. How many red faces are there on the second cube?

The colour on the top of the first cube is irrelevant. Once it is rolled, we must have three red faces and three blue faces on the second cube if the probability of like faces is $\frac{1}{2}$.

The average score was 2.7.

6. The triangle ABC has sides $AB = 137$, $AC = 241$, and $BC = 200$. There is a point D, on BC, such that both incircles of triangles ABD and ACD touch AD at the same point E. Determine the length of CD.



The solution to this problem is obtained from using the property that tangents to a circle from an external point are equal. Using this fact and applying algebraic variables as needed, one obtains, from the resulting equation, $CD = 152$.

The average score was 0.5.

7. Determine the minimum value of $f(x)$ where

$$f(x) = (3\sin x - 4\cos x - 10)(3\sin x + 4\cos x - 10).$$

While calculus can be used, it is not necessary. Multiply the given expressions together and substitute for $\cos^2 x$. The result is a quadratic in $\sin x$. Completing the square and noting that $|\sin x| \leq 1$ yields a minimum value of 49.

The average score was 1.2.

8. An hourglass is formed from two identical cones. Initially, the upper cone is filled with sand and the lower one is empty. The sand flows at a constant rate from the upper to the lower cone. It takes exactly one hour to empty the upper cone. How long does it take for the depth of sand in the lower cone to be half the depth of sand in the upper cone? (Assume that the sand stays level in both cones at all times.)

At the required time the depth of sand in the upper cone is two-thirds its original depth. Since volume varies as the cube of any dimension in regular figures, the time required is $1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}$ of an hour.

The average score was 0.7.

Part B

Note: All questions in part B were graded out of 10 points.

1. The straight line l_1 with equation $x - 2y + 10 = 0$ meets the circle with equation $x^2 + y^2 = 100$ at B in the first quadrant. A line through B, perpendicular to l_1 cuts the y-axis at $P(0, t)$. Determine the value of t.

This is a straightforward problem. Solve for point B by substitution from the first equation to the second equation to obtain B as $(6, 8)$. The line through B perpendicular to l_1 meets the y-axis at $(0, 20)$.

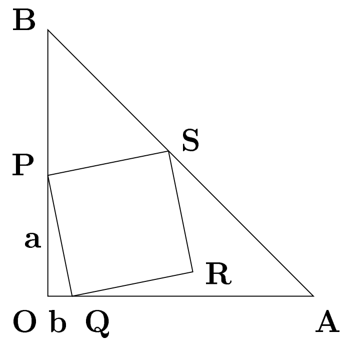
The average score was 5.9

2. Consider the ten numbers $ar, ar^2, ar^3, \cdots, ar^{10}$. If their sum is 18 and the sum of their reciprocals is 6, determine their product.

Consider the given equations, and divide the first by the second. This yields $a^2r^{11} = 3$. The required result is then $a^{10}r^{55} = 3^5$.

The average score was 2.1.

3. In an isosceles right-angled triangle AOB, points P, Q and S are chosen on sides OB, OA and AB respectively such that a square $PQRS$ is formed as shown. If the lengths of OP and OQ are a and b respectively, and the area of $PQRS$ is $\frac{2}{5}$ that of triangle AOB, determine $a : b$.



One method is to draw ST perpendicular to OB. Congruent triangles are obtained, yielding $OB = 2a + b$. Another method is to use the sine and cosine laws on $\triangle BPS$, and to use analytic geometry. The result is $a : b = 2 : 1$.

The average score was 0.7.

4. Find all real values of x, y and z such that

$$\begin{aligned} x - \sqrt{yz} &= 42 \\ y - \sqrt{xz} &= 6 \\ z - \sqrt{xy} &= -30. \end{aligned}$$

Let $x = a^2,\ y = b^2,\ z = c^2$, thereby eliminating the radicals. Combining the equations in pairs leads to the fact that $b = \frac{a+c}{2}$. This allows a reduction from three to two variables, and hence the result $x = 54,\ y = 24,\ z = 6$.

The average score was 0.4.