Draft Solutions for 2016 CMO – April 27, 2016

1. The integers $1, 2, 3, \ldots, 2016$ are written on a board. You can choose any two numbers on the board and replace them with their average. For example, you can replace 1 and 2 with 1.5, or you can replace 1 and 3 with a second copy of 2. After 2015 replacements of this kind, the board will have only one number left on it.

   (a) Prove that there is a sequence of replacements that will make the final number equal to 2.

   (b) Prove that there is a sequence of replacements that will make the final number equal to 1000.

   Solution: (a) First replace 2014 and 2016 with 2015, and then replace the two copies of 2015 with a single copy. This leaves us with $\{1, 2, \ldots, 2013, 2015\}$. From here, we can replace 2013 and 2015 with 2014 to get $\{1, 2, \ldots, 2012, 2014\}$. We can then replace 2012 and 2014 with 2013, and so on, until we eventually get to $\{1, 3\}$. We finish by replacing 1 and 3 with 2.

   (b) Using the same construction as in (a), we can find a sequence of replacements that reduces $\{a, a+1, \ldots, b\}$ to just $\{a+1\}$. Similarly, can also find a sequence of replacements that reduces $\{a, a+1, \ldots, b\}$ to just $\{b-1\}$.

   In particular, we can find sequences of replacements that reduce $\{1, 2, \ldots, 999\}$ to just $\{998\}$, and that reduce $\{1001, 1002, \ldots, 2016\}$ to just $\{1002\}$. This leaves us with $\{998, 1000, 1002\}$. We can replace 998 and 1002 with a second copy of 1000, and then replace the two copies of 1000 with a single copy to complete the construction.

2. Consider the following system of 10 equations in 10 real variables $v_1, \ldots, v_{10}$:

$$v_i = 1 + \frac{6\,v_i^2}{v_1^2 + v_2^2 + \cdots + v_{10}^2} \qquad (i = 1, \ldots, 10).$$

   Find all 10-tuples $(v_1, v_2, \ldots, v_{10})$ that are solutions of this system.

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Solution:
For a particular solution $(v_1, v_2, \dots, v_{10})$, let $s = v_1^2 + v_2^2 + \cdots + v_{10}^2$. Then

$$v_i = 1 + \frac{6v_i^2}{s} \quad \Rightarrow \quad 6v_i^2 - sv_i + s = 0.$$

Let a and b be the roots of the quadratic $6x^2 - sx + s = 0$, so for each i, $v_i = a$ or $v_i = b$. We also have $ab = s/6$ (by Vieta's formula, for example).

If all the v_i are equal, then

$$v_i = 1 + \frac{6}{10} = \frac{8}{5}$$

for all i. Otherwise, let $5 + k$ of the v_i be a, and let $5 - k$ of the v_i be b, where $0 < k \leq 4$. Then by the AM-GM inequality,

$$6ab = s = (5+k)a^2 + (5-k)b^2 \geq 2ab\sqrt{25 - k^2}.$$

From the given equations, $v_i \geq 1$ for all i, so a and b are positive. Then $\sqrt{25-k^2} \leq 3 \Rightarrow 25 - k^2 \leq 9 \Rightarrow k^2 \geq 16 \Rightarrow k = 4$. Hence, $6ab = 9a^2 + b^2 \Rightarrow (b - 3a)^2 = 0 \Rightarrow b = 3a$.

Adding all given ten equations, we get

$$v_1 + v_2 + \cdots + v_{10} = 16.$$

But $v_1 + v_2 + \cdots + v_{10} = 9a + b = 12a$, so $a = 16/12 = 4/3$ and $b = 4$. Therefore, the solutions are $(8/5, 8/5, \dots, 8/5)$ and all ten permutations of $(4/3, 4/3, \dots, 4/3, 4)$.

3. Find all polynomials $P(x)$ with integer coefficients such that $P(P(n)+n)$ is a prime number for infinitely many integers n.

Answer: $P(n) = p$ where p is a prime number and $P(n) = -2n + b$ where b is odd.

Solution: Note that if $P(n) = 0$ then $P(P(n) + n) = P(n) = 0$ which is not prime. Let $P(x)$ be a degree k polynomial of the form $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_0$ and note that if $P(n) \neq 0$ then

$$\begin{aligned} P(P(n)+n) - P(n) &= \\ &a_k[(P(n)+n)^k - n^k] + a_{k-1}[(P(n)+n)^{k-1} - n^{k-1}] + \cdots + a_1 P(n) \end{aligned}$$

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which is divisible by $(P(n)+n)-n = P(n)$. Therefore if $P(P(n)+n)$ is prime then either $P(n) = \pm 1$ or $P(P(n)+n) = \pm P(n) = p$ for some prime number p. Since $P(x)$ is a polynomial, it follows that $P(n) = \pm 1$ for only finitely many integers n. Therefore either $P(n) = P(P(n)+n)$ for infinitely many integers n or $P(n) = -P(P(n)+n)$ for infinitely many integers n. Suppose that $P(n) = P(P(n)+n)$ for infinitely many integers n. This implies that the polynomial $P(P(x)+x) - P(x)$ has infinitely many roots and thus is identically zero. Therefore $P(P(x)+x) = P(x)$ holds identically. Now note that if $k \geq 2$ then $P(P(x)+x)$ has degree k^2 while $P(x)$ has degree k, which is not possible. Therefore $P(x)$ is at most linear with $P(x) = ax+b$ for some integers a and b. Now note that

$$P(P(x)+x) = a(a+1)x + ab + b$$

and thus $a = a(a+1)$ and $ab+b = b$. It follows that $a = 0$ which leads to the solution $P(n) = p$ where p is a prime number. By the same argument if $P(n) = -P(P(n)+n)$ for infinitely many integers n then $P(x) = -P(P(x)+x)$ holds identically and $P(x)$ is linear with $P(x) = ax+b$. In this case it follows that $a = -a(a+1)$ and $ab+b = -b$. This implies that either $a = 0$ or $a = -2$. If $a = -2$ then $P(n) = -2n+b$ which is prime for some integers n only if b is odd. Note that in this case $P(P(n)+n) = 2n-b$ which is indeed prime for infinitely many integers n as long as b is odd. $\square$

4. Lavaman versus the Flea. Let A, B, and F be positive integers, and assume $A < B < 2A$. A flea is at the number 0 on the number line. The flea can move by jumping to the right by A or by B. Before the flea starts jumping, Lavaman chooses finitely many intervals $\{m+1, m+2, \ldots, m+A\}$ consisting of A consecutive positive integers, and places lava at all of the integers in the intervals. The intervals must be chosen so that:

(i) any two distinct intervals are disjoint and not adjacent;
(ii) there are at least F positive integers with no lava between any two intervals; and
(iii) no lava is placed at any integer less than F.

Prove that the smallest F for which the flea can jump over all the intervals and avoid all the lava, regardless of what Lavaman does,

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is $F = (n-1)A + B$, where n is the positive integer such that $\dfrac{A}{n+1} \le B - A < \dfrac{A}{n}$.

Solution: Let $B = A + C$ where $A/(n+1) \le C < A/n$.

First, here is an informal sketch of the proof.

Lavaman's strategy: Use only safe intervals with $nA + C - 1$ integers. The flea will start at position $[1, C]$ from the left, which puts him at position $[nA, nA + C - 1]$ from the right. After $n - 1$ jumps, he will still have $nA - (n-1)(A + C) = A - (n-1)C > C$ distance to go, which is not enough for a big jump to clear the lava. Thus, he must do at least n jumps in the safe interval, but that's possible only with all small jumps, and furthermore is impossible if the starting position is C. This gives him starting position 1 higher in the next safe interval, so sooner or later the flea is going to hit the lava.

Flea's strategy: The flea just does one interval at a time. If the safe interval has at least $nA+C$ integers in it, the flea has distance $d > nA$ to go to the next lava when it starts. Repeatedly do big jumps until d is between 1 and $C \bmod A$, then small jumps until the remaining distance is between 1 and C, then a final big jump. This works as long as the first part does. However, we get at least n big jumps since floor($(d-1)/A$) can never go down two from a big jump (or we'd be done doing big jumps), so we get n big jumps, and thus we are good if $d \bmod A$ is in any of $[1, C]$, $[C + 1, 2C]$, $\ldots [nC + 1, (n + 1)C]$, but that's everything. □

Let $C = B - A$. We shall write our intervals of lava in the form $(L_i, R_i] = \{L_i + 1, L_i + 2, \ldots, R_i\}$, where $R_i = L_i + A$ and $R_{i-1} < L_i$ for every $i \ge 1$. We also let $R_0 = 0$. We shall also represent a path for the flea as a sequence of integers $x_0, x_1, x_2, \ldots$ where $x_0 = 0$ and $x_j - x_{j-1} \in \{A, B\}$ for every $j \ge 0$.

Now here is a detailed proof.

First, assume $F < (n-1)A + B$ $(= nA + C)$: we must prove that Lavaman has a winning strategy. Let $L_i = R_{i-1} + nA + C - 1$ for every $i \ge 1$. (Observe that $nA + C - 1 \ge F$.)

Assume that the flea has an infinite path that avoids all the lava, which

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means that $x_j \notin (L_i, R_i]$ for all $i, j \geq 1$. For each $i \geq 1$, let

$$M_i = \max\{x_j : x_j \leq L_i\}, \quad m_i = \min\{x_j : x_j > R_i\}, \quad \text{and} \quad J(i) = \max\{j : x_j \leq L_i\}.$$

Also let $m_0 = 0$. Then for $i \geq 1$ we have

$$M_i = x_{J(i)} \quad \text{and} \quad m_i = x_{J(i)+1}.$$

Also, for every $i \geq 1$, we have
(a) $m_i = M_i + B$ (because $M_i + A \leq L_i + A = R_i$);
(b) $L_i \geq M_i > L_i - C$ (since $M_i = m_i - B > R_i - B = L_i + A - B$); and
(c) $R_i < m_i \leq R_i + C$ (since $m_i = M_i + B \leq L_i + B = R_i + C$).

<u>Claim 1</u>: $J(i+1) = J(i) + n + 1$ for every $i \geq 1$. (That is, after jumping over one interval of lava, the flea must make exactly n jumps before jumping over the next interval of lava.)
Proof:

$$\begin{aligned} x_{J(i)+n+1} &\leq x_{J(i)+1} + Bn \\ &= m_i + Bn \\ &< R_i + C + \left(A + \frac{A}{n}\right) n \\ &= L_{i+1} + A + 1. \end{aligned}$$

Because of the strict inequality, we have $x_{J(i)+n+1} \leq R_{i+1}$, and hence $x_{J(i)+n+1} \leq L_{i+1}$. Therefore $J(i) + n + 1 \leq J(i+1)$. Next, we have

$$\begin{aligned} x_{J(i)+n+1} &\geq x_{J(i)+1} + An \\ &= m_i + An \\ &> R_i + An \\ &= L_{i+1} - C + 1 \\ &> L_{i+1} - A + 1 \quad (\text{since } C < A). \end{aligned}$$

Therefore $x_{J(i)+n+2} \geq x_{J(i)+n+1} + A > L_{i+1}$, and hence $J(i+1) < J(i) + n + 2$. Claim 1 follows.

<u>Claim 2</u>: $x_{j+1} - x_j = A$ for all $j = J(i) + 1, \ldots, J(i+1) - 1$, for all $i \geq 1$. (That is, the n intermediate jumps of Claim 1 must all be of

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length A.)
Proof: If Claim 2 is false, then

$$\begin{aligned} M_{i+1} = x_{J(i+1)} = x_{J(i)+n+1} &\geq x_{J(i)+1} + (n-1)A + B \\ &> R_i + nA + C \\ &= L_{i+1} + 1 \\ &> M_{i+1} \end{aligned}$$

which is a contradiction. This proves Claim 2.

We can now conclude that

$$x_{J(i+1)+1} = x_{J(i)+n+2} = x_{J(i)+1} + nA + B\,;$$

$$\text{i.e.,} \qquad m_{i+1} = m_i + nA + B \qquad \text{for each } i \geq 1.$$

Therefore

$$\begin{aligned} m_{i+1} - R_{i+1} &= m_i + nA + B - (R_i + nA + C - 1 + A) \\ &= m_i - R_i + 1\,. \end{aligned}$$

Hence

$$C \geq m_{C+1} - R_{C+1} = m_1 - R_1 + C > C$$

which is a contradiction. Therefore no path for the flea avoids all the lava. We observe that Lavaman only needs to put lava on the first $C+1$ intervals.

Now assume $F \geq (n-1)A + B$. We will show that the flea can avoid all the lava. We shall need the following result:

<u>Claim 3</u>: Let $d \geq nA$. Then there exist nonnegative integers s and t such that $sA + tB \in (d-C, d]$.

We shall prove this result at the end.

First, observe that $L_1 \geq nA$. By Claim 3, it is possible for the flea to make a sequence of jumps starting from 0 and ending at a point of $(L_1 - C, L_1]$. From any point of this interval, a single jump of size B takes the flea over $(L_1, R_]$ to a point in $(R_1, R_1 + C]$, which corresponds to the point $x_{J(1)+1}$ $(= m_1)$ on the flea's path.

Now we use induction to prove that, for every $i \geq 1$, there is a path such that x_j avoids lava for all $j \leq J(i) + 1$. The case $i = 1$ is done, so

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assume that the assertion holds for a given i. Then $x_{J(i)+1} = m_i \in (R_i, R_i + C]$. Therefore

$$L_{i+1} - m_i \ \geq\ R_i + F - (R_i + C) \ =\ F - C \ \geq\ nA\,.$$

Applying Claim 3 with $d = L_{i+1} - m_i$ shows that the flea can jump from m_i to a point of $(L_{i+1} - C, L_{i+1}]$. A single jump of size B then takes the flea to a point of $(R_{i+1}, R_{i+1} + C]$ (without visiting $(L_{i+1}, R_{i+1}]$), and this point serves as $x_{J(i+1)+1}$. This completes the induction.

Proof of Claim 3: Let u be the greatest integer that is less than or equal to d/A. Then $u \geq n$ and $uA \leq d < (u+1)A$. For $v = 0, \ldots, n$, let

$$z_v \ =\ (u-v)A + vB \ =\ uA + vC\,.$$

Then

$$\begin{aligned} z_0 \ &=\ uA \ \leq\ d, \\ z_n \ &=\ uA + nC \ =\ uA + (n+1)C - C \ \geq\ (u+1)A - C \ >\ d - C\,. \end{aligned}$$

and $z_{v+1} - z_v \ =\ C \quad$ for $v = 0, \ldots, n-1$.

Therefore we must have $z_v \in (d - C, d]$ for some v in $\{0, 1, \ldots, n\}$. $\square$

5. Let $\triangle ABC$ be an acute-angled triangle with altitudes AD and BE meeting at H. Let M be the midpoint of segment AB, and suppose that the circumcircles of $\triangle DEM$ and $\triangle ABH$ meet at points P and Q with P on the same side of CH as A. Prove that the lines ED, PH, and MQ all pass through a single point on the circumcircle of $\triangle ABC$.

Solution:

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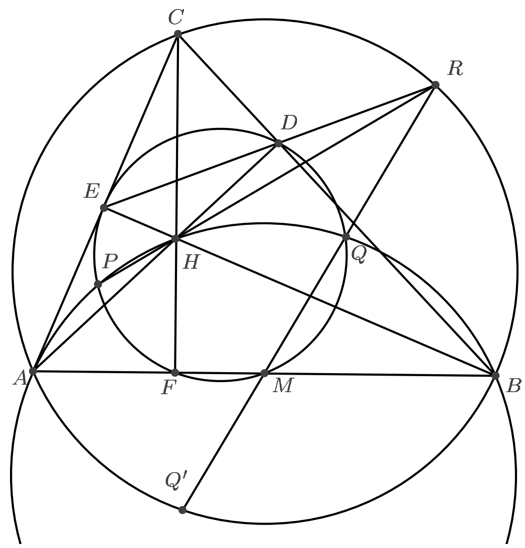

Let R denote the intersection of lines ED and PH. Since quadrilaterals $ECDH$ and $APHB$ are cyclic, we have $\angle RDA = 180° - \angle EDA = 180° - \angle EDH = 180° - \angle ECH = 90° + A$, and $\angle RPA = \angle HPA = 180° - \angle HBA = 90° + A$. Therefore, $APDR$ is cyclic. This in turn implies that $\angle PBE = \angle PBH = \angle PAH = \angle PAD = \angle PRD = \angle PRE$, and so $PBRE$ is also cyclic.

Let F denote the base of the altitude from C to AB. Then D, E, F, and M all lie on the 9-point circle of $\triangle ABC$, and so are cyclic. We also know $APDR$, $PBRE$, $BCEF$, and $ACDF$ are cyclic, which implies $\angle ARB = \angle PRB - \angle PRA = \angle PEB - \angle PDA = \angle PEF + \angle FEB - \angle PDF + \angle ADF = \angle FEB + \angle ADF = \angle FCB + \angle ACF = C$. Therefore, R lies on the circumcircle of $\triangle ABC$.

Now let Q' and R' denote the intersections of line MQ with the circumcircle of $\triangle ABC$, chosen so that Q', M, Q, R' lie on the line in that order. We will show that $R' = R$, which will complete the proof. However, first note that the circumcircle of $\triangle ABC$ has radius $\frac{AB}{2\sin C}$, and the circumcircle of $\triangle ABH$ has radius $\frac{AB}{2\sin\angle AHB} = \frac{AB}{2\sin(180° - C)}$. Thus the two circles have equal radius, and so they must be symmetrical about the point M. In particular, $MQ = MQ'$.

Since $\angle AEB = \angle ADB = 90°$, we furthermore know that M is the circumcenter of both $\triangle AEB$ and $\triangle ADB$. Thus, $MA = ME = MD = MB$. By Power of a Point, we then have $MQ \cdot MR' = MQ' \cdot MR' = MA \cdot MB = MD^2$. In particular, this means that the circumcircle of

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$\triangle DR'Q$ is tangent to MD at D, which means $\angle MR'D = \angle MDQ$. Similarly $MQ \cdot MR' = ME^2$, and so $\angle MR'E = \angle MEQ = \angle MDQ = \angle MR'D$. Therefore, R' also lies on the line ED.

Finally, the same argument shows that MP also intersects the circumcircle of $\triangle ABC$ at a point R'' on line ED. Thus, R, R', and R'' are all chosen from the intersection of the circumcircle of $\triangle ABC$ and the line ED. In particular, two of R, R', and R'' must be equal. However, $R'' \neq R$ since MP and PH already intersect at P, and $R'' \neq R'$ since MP and MQ already intersect at M. Thus, $R' = R$, and the proof is complete. $\square$