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45^{th} Canadian Mathematical Olympiad

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CMS
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Problems and Solutions

1. Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

Solution 1: The answer is $P(x)$ being any constant polynomial and $P(x) \equiv kx^2+kx+c$ for any (nonzero) constant k and constant c.

Let Λ be the expression $(x+1)P(x-1)-(x-1)P(x)$, i.e. the expression in the problem statement.

Substituting $x=-1$ into Λ yields $2P(-1)$ and substituting $x=1$ into Λ yield $2P(0)$. Since $(x+1)P(x-1)-(x-1)P(x)$ is a constant polynomial, $2P(-1)=2P(0)$. Hence, $P(-1)=P(0)$.

Let $c=P(-1)=P(0)$ and $Q(x)=P(x)-c$. Then $Q(-1)=Q(0)=0$. Hence, $0,-1$ are roots of $Q(x)$. Consequently, $Q(x)=x(x+1)R(x)$ for some polynomial R. Then $P(x)-c=x(x+1)R(x)$, or equivalently, $P(x)=x(x+1)R(x)+c$.

Substituting this into Λ yield

$$(x+1)((x-1)xR(x-1)+c)-(x-1)(x(x+1)R(x)+c)$$

This is a constant polynomial and simplifies to

$$x(x-1)(x+1)(R(x-1)-R(x))+2c.$$

Since this expression is a constant, so is $x(x-1)(x+1)(R(x-1)-R(x))$. Therefore, $R(x-1)-R(x)=0$ as a polynomial. Therefore, $R(x)=R(x-1)$ for all $x \in \mathbb{R}$. Then $R(x)$ is a polynomial that takes on certain values for infinitely values of x. Let k be such a value. Then $R(x)-k$ has infinitely many roots, which can occur if and only if $R(x)-k=0$. Therefore, $R(x)$ is identical to a constant k. Hence, $Q(x)=kx(x+1)$ for some constant k. Therefore, $P(x)=kx(x+1)+c=kx^2+kx+c$.

Finally, we verify that all such $P(x)=kx(x+1)+c$ work. Substituting this into Λ yield

$$\begin{aligned} & (x+1)(kx(x-1)+c)-(x-1)(kx(x+1)+c) \\ = \; & kx(x+1)(x-1)+c(x+1)-kx(x+1)(x-1)-c(x-1)=2c. \end{aligned}$$

Hence, $P(x)=kx(x+1)+c=kx^2+kx+c$ is a solution to the given equation for any constant k. Note that this solution also holds for $k=0$. Hence, constant polynomials are also solutions to this equation. $\square$

Solution 2: As in Solution 1, any constant polynomial P satisfies the given property. Hence, we will assume that P is not a constant polynomial.

Let n be the degree of P. Since P is not constant, $n \geq 1$. Let

$$P(x)=\sum_{i=0}^{n} a_i x^i,$$

with $a_n \neq 0$. Then

$$(x+1)\sum_{i=0}^{n} a_i (x-1)^i-(x-1)\sum_{i=0}^{n} a_i x^i=C,$$

for some constant C. We will compare the coefficient of x^n of the left-hand side of this equation with the right-hand side. Since C is a constant and $n \geq 1$, the coefficient of x^n of the right-hand side is equal to zero. We now determine the coefficient of x^n of the left-hand side of this expression.

The left-hand side of the equation simplifies to

$$x\sum_{i=0}^{n} a_i (x-1)^i+\sum_{i=0}^{n} a_i (x-1)^i-x\sum_{i=0}^{n} a_i x^i+\sum_{i=0}^{n} a_i x^i.$$

We will determine the coefficient x^n of each of these four terms.

By the Binomial Theorem, the coefficient of x^n of the first term is equal to that of $x\left(a_{n-1}(x-1)^{n-1}+a_n(x-1)^n\right)=a_{n-1}-\binom{n}{n-1}a_n=a_{n-1}-na_n$.

The coefficient of x^n of the second term is equal to that of $a_n(x-1)^n$, which is a_n.

The coefficient of x^n of the third term is equal to a_{n-1} and that of the fourth term is equal to a_n.

Summing these four coefficients yield $a_{n-1}-na_n+a_n-a_{n-1}+a_n=(2-n)a_n$.

This expression is equal to 0. Since $a_n \neq 0$, $n=2$. Hence, P is a quadratic polynomial.

Let $P(x)=ax^2+bx+c$, where a,b,c are real numbers with $a\neq 0$. Then

$$(x+1)(a(x-1)^2+b(x-1)+c)-(x-1)(ax^2+bx+c)=C.$$

Simplifying the left-hand side yields

$$(b-a)x+2c=2C.$$

Therefore, $b-a=0$ and $2c=2C$. Hence, $P(x)=ax^2+ax+c$. As in Solution 1, this is a valid solution for all $a\in\mathbb{R}\backslash\{0\}$. $\square$

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2. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that $0, a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_n$ all have different remainders when divided by $n + 1$?

Solution: It is possible if and only if n is odd.

If n is even, then $a_1 + a_2 + \ldots + a_n = 1 + 2 + \ldots + n = \frac{n}{2} \cdot (n + 1)$, which is congruent to 0 mod $n + 1$. Therefore, the task is impossible.

Now suppose n is odd. We will show that we can construct $a_1, a_2, \ldots, a_n$ that satisfy the conditions given in the problem. Then let $n = 2k + 1$ for some non-negative integer k. Consider the sequence: $1, 2k, 3, 2k - 2, 5, 2k - 3, \ldots, 2, 2k + 1$, i.e. for each $1 \le i \le 2k + 1$, $a_i = i$ if i is odd and $a_i = 2k + 2 - i$ if i is even.

We first show that each term $1, 2, \ldots, 2k + 1$ appears exactly once. Clearly, there are $2k + 1$ terms. For each odd number m in $\{1, 2, \ldots, 2k + 1\}$, $a_m = m$. For each even number m in this set, $a_{2k+2-m} = 2k + 2 - (2k + 2 - m) = m$. Hence, every number appears in $a_1, \ldots, a_{2k+1}$. Hence, $a_1, \ldots, a_{2k+1}$ does consist of the numbers $1, 2, \ldots, 2k + 1$ in some order.

We now determine $a_1 + a_2 + \ldots + a_m \pmod{2k + 2}$. We will consider the cases when m is odd and when m is even separately. Let $b_m = a_1 + a_2 \ldots + a_m$.

If m is odd, note that $a_1 \equiv 1 \pmod{2k + 2}$, $a_2 + a_3 = a_4 + a_5 = \ldots = a_{2k} + a_{2k+1} = 2k + 3 \equiv 1 \pmod{2k + 2}$. Therefore, $\{b_1, b_3, \ldots, b_{2k+1}\} = \{1, 2, 3, \ldots, k + 1\} \pmod{2k + 2}$.

If m is even, note that $a_1 + a_2 = a_3 + a_4 = \ldots = a_{2k-1} + a_{2k} = 2k + 1 \equiv -1 \pmod{2k + 2}$. Therefore, $\{b_2, b_4, \ldots, b_{2k}\} = \{-1, -2, \ldots, -k\} \pmod{2k + 2} \equiv \{2k + 1, 2k, \ldots, k + 2\} \pmod{2k + 2}$.

Therefore, $b_1, b_2, \ldots, b_{2k+1}$ do indeed have different remainders when divided by $2k + 2$. This completes the problem. $\square$

3. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

Solution 1. Since $\angle C = 90^\circ$, the point C lies on the semicircle with diameter AB which implies that, if M is te midpoint of side AB, then $MA = MC = MB$. This implies that triangle AMC is isosceles and hence that $\angle ACM = \angle A$. By definition, G lies on segment M and it follows that $\angle ACG = \angle ACM = \angle A = \angle CPA$. This implies that triangles APC and ACG are similar and hence that $AC^2 = AG \cdot AP$. Now if D denotes the foot of the perpendicular from C to AB, it follows that triangles ACD and ABC are similar which implies that $AC^2 = AD \cdot AB$. Therefore $AG \cdot AP = AC^2 = AD \cdot AB$ and, by power of a point, quadrilateral $DGPB$ is cyclic. This implies that D lies on the circumcircle of triangle BPG and, by a symmetric argument, it follows that D also lies on the circumcircle of triangle AGQ. Therefore these two circumcircles meet at the point D on side AB.

Solution 2. Define D and M as in Solution 1. Let R be the point on side AB such that $AC = CR$ and triangle ACR is isosceles. Since $\angle CRA = \angle A = \angle CPA$, it follows that $CPRA$ is cyclic and hence that $\angle GPR = \angle APR = \angle ACR = 180^\circ - 2\angle A$. As in Solution 1, $MC = MB$ and hence $\angle GMR = \angle CMB = 2\angle A = 180^\circ - \angle GPR$. Therefore $GPRM$ is cyclic and, by power of a point, $AM \cdot AR = AG \cdot AP$. Since ACR is isosceles, D is the midpoint of AR and thus, since M is the midpoint of AB, it follows that $AM \cdot AR = AD \cdot AB = AG \cdot AP$. Therefore $DGPB$ is cyclic, implying the result as in Solution 1.

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4. Let n be a positive integer. For any positive integer j and positive real number r, define

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \quad \text{and} \quad g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \leq n^2 + n \leq \sum_{j=1}^{n} g_j(r).$$

Solution 1: We first prove the left hand side inequality. We begin by drawing an $n \times n$ board, with corners at $(0,0), (n,0), (0,n)$ and (n,n) on the Cartesian plane.

Consider the line ℓ with slope r passing through $(0,0)$. For each $j \in \{1, \ldots, n\}$, consider the point $(j, \min(jr, n))$. Note that each such point either lies on ℓ or the top edge of the board. In the j^{th} column from the left, draw the rectangle of height $\min(jr, n)$. Note that the sum of the n rectangles is equal to the area of the board under the line ℓ plus n triangles (possibly with area 0) each with width at most 1 and whose sum of the heights is at most n. Therefore, the sum of the areas of these n triangles is at most $n/2$. Therefore, $\sum_{j=1}^{n} \min(jr, n)$ is at most the area of the square under ℓ plus $n/2$.

Consider the line with slope $1/r$. By symmetry about the line $y = x$, the area of the square under the line with slope $1/r$ is equal to the area of the square above the line ℓ. Therefore, using the same reasoning as before, $\sum_{j=1}^{n} \min(j/r, n)$ is at most the area of the square above ℓ plus $n/2$.

Therefore, $\sum_{j=1}^{n} f_j(r) = \sum_{j=1}^{n} (\min(jr, n) + \min(\frac{j}{r}, n))$ is at most the area of the board plus n, which is $n^2 + n$. This proves the left hand side inequality.

To prove the right hand side inequality, we will use the following lemma:

Lemma: Consider the line ℓ with slope s passing through $(0,0)$. Then the number of squares on the board that contain an interior point below ℓ is $\sum_{j=1}^{n} \min(\lceil js \rceil, n)$.

Proof of Lemma: For each $j \in \{1, \ldots, n\}$, we count the number of squares in the j^{th} column (from the left) that contain an interior point lying below the line ℓ. The line $x = j$ intersect the line ℓ at (j, js). Hence, since each column contains n squares total, the number of such squares is $\min(\lceil js \rceil, n)$. Summing over all $j \in \{1, 2, \ldots, n\}$ proves the lemma. *End Proof of Lemma*

By the lemma, the rightmost expression of the inequality is equal to the number of squares containing an interior point below the line with slope r plus the number of squares containing an interior point below the line with slope $1/r$. By symmetry about the line $y = x$, the latter number is equal to the number of squares containing an interior point above the line with slope r. Therefore, the rightmost expression of the inequality is equal to the number of squares of the board plus the number of squares of which ℓ passes through the interior. The former is equal to n^2. Hence, to prove the inequality, it suffices to show that every line passes through the interior of at least n squares. Since ℓ has positive slope, each ℓ passes through either n rows and/or n columns. In either case, ℓ passes through the interior of at least n squares. Hence, the right inequality holds. □

Solution 2: We first prove the left inequality. Define the function $f(r) = \sum_{j=1}^{n} f_j(r)$. Note that $f(r) = f(1/r)$ for all $r > 0$. Therefore, we may assume that $r \geq 1$.

Let $m = \lfloor n/r \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Then $\min(jr, n) = jr$ for all $j \in \{1, \ldots, m\}$ and $\min(jr, n) = n$ for all $j \in \{m+1, \ldots, n\}$. Note that since $r \geq 1$, $\min(j/r, n) \leq n$ for all $j \in \{1, \ldots, n\}$. Therefore,

$$f(r) = \sum_{j=1}^{n} f_j(r) = (1 + 2 + \ldots m)r + (n - m)n + (1 + 2 + \ldots + n) \cdot \frac{1}{r}$$

$$= \frac{m(m+1)}{2} \cdot r + \frac{n(n+1)}{2} \cdot \frac{1}{r} + n(n-m) \tag{1}$$

Then by (**??**), note that $f(r) \leq n^2 + n$ if and only if

$$\frac{m(m+1)r}{2} + \frac{n(n+1)}{2r} \leq n(m+1)$$

if and only if

$$m(m+1)r^2 + n(n+1) \leq 2rn(m+1) \tag{2}$$

Since $m = \lfloor n/r \rfloor$, there exist a real number b satisfying $0 \leq b < r$ such that $n = mr + b$. Substituting this into (**??**) yields

$$m(m+1)r^2 + (mr + b)(mr + b + 1) \leq 2r(mr + b)(m + 1),$$

if and only if

$$2m^2r^2 + mr^2 + (2mb + m)r + b^2 + b \leq 2m^2r^2 + 2mr^2 + 2mbr + 2br,$$

which simplifies to $mr + b^2 + b \leq mr^2 + 2br \Leftrightarrow b(b + 1 - 2r) \leq mr(r - 1) \Leftrightarrow b((b - r) + (1 - r)) \leq mr(r - 1)$. This is true since

$$b((b - r) + (1 - r)) \leq 0 \leq mr(r - 1),$$

which holds since $r \geq 1$ and $b < r$. Therefore, the left inequality holds.

We now prove the right inequality. Define the function $g(r) = \sum_{j=1}^{n} = g_j(r)$. Note that $g(r) = g(1/r)$ for all $r > 0$. Therefore, we may assume that $r \geq 1$. We will consider two cases; $r \geq n$ and $1 \leq r < n$.

If $r \geq n$, then $\min(\lceil jr \rceil, n) = n$ and $\min(\lceil j/r \rceil, n) = 1$ for all $j \in \{1, \ldots, n\}$. Hence, $g_j(r) = n + 1$ for all $j \in \{1, \ldots, n\}$. Therefore, $g(r) = n(n + 1) = n^2 + n$, implying that the inequality is true.

Now we consider the case $1 \leq r < n$. Let $m = \lfloor n/r \rfloor$. Hence, $jr \leq n$ for all $j \in \{1, \ldots, m\}$, i.e. $\min(\lceil jr, \rceil, n) = \lceil jr \rceil$ and $jr \geq n$ for all $j \in \{m + 1, \ldots, n\}$, i.e. $\min(\lceil jr \rceil, n) = n$. Therefore,

$$\sum_{j=1}^{n} \min(\lceil jr \rceil, n) = \sum_{j=1}^{m} \lceil jr \rceil + (n - m)n. \tag{3}$$

We will now consider the second sum $\sum_{j=1}^{n} \min\{\lceil j/r \rceil, n\}$.

Since $r \geq 1$, $\min(\lceil j/r \rceil, n) \leq \min(\lceil n/r \rceil, n) \leq n$. Therefore, $\min(\lceil j/r \rceil, n) = \lceil j/r \rceil$. Since $m = \lfloor n/r \rfloor$, $\lceil n/r \rceil \leq m + 1$. Since $r > 1$, $m < n$, which implies that $m + 1 \leq n$. Therefore, $\min\{\lceil j/r \rceil, n\} = \lceil j/r \rceil \leq \lceil n/r \rceil \leq m + 1$ for all $j \in \{1, \ldots, n\}$.

For each positive integer $k \in \{1, \ldots, m + 1\}$, we now determine the number of positive integers $j \in \{1, \ldots, n\}$ such that $\lceil j/r \rceil = k$. We denote this number by s_k.

Note that $\lceil j/r \rceil = k$ if and only if $k - 1 < j/r \leq k$ if and only if $(k - 1)r < j \leq \min(kr, n)$, since $j \leq n$. We will handle the cases $k \in \{1, \ldots, m\}$ and $k = m + 1$ separately. If $k \in \{1, \ldots, m\}$, then $\min(kr, n) = kr$, since $r \leq m$ and $m = \lfloor n/r \rfloor$.

The set of positive integers j satisfying $(k-1)r < j \leq kr$ is $\{\lfloor (k-1)r \rfloor + 1, \lfloor (k-1)r \rfloor + 2, \ldots, \lfloor kr \rfloor\}$. Hence,

$$s_k = \lfloor rk \rfloor - (\lfloor r(k-1) \rfloor + 1) + 1 = \lfloor rk \rfloor - \lfloor r(k-1) \rfloor$$

for all $k \in \{1, \ldots, m\}$. If $k = m+1$, then $(k-1)r < j \leq \min(kr, n) = n$. The set of positive integers j satisfying $(k-1)r < j \leq kr$ is $\{\lfloor (k-1)r \rfloor + 1, \ldots, n\}$. Then $s_{m+1} = n - \lfloor r(k-1) \rfloor = n - \lfloor mr \rfloor$. Note that this number is non-negative by the definition of m. Therefore, by the definition of s_k, we have

$$\begin{aligned} \sum_{j=1}^{n} \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right) &= \sum_{k=1}^{m+1} k s_k \\ &= \sum_{k=1}^{m} \left(k\left(\lfloor kr \rfloor - \lfloor (k-1)r \rfloor\right)\right) + (m+1)(n - \lfloor rm \rfloor) = (m+1)n - \sum_{k=1}^{m} \lfloor kr \rfloor. \end{aligned} \tag{4}$$

Summing (**??**) and (**??**) yields that

$$g(r) = n^2 + n + \sum_{j=1}^{m} \left(\lceil jr \rceil - \lfloor jr \rfloor\right) \geq n^2 + n,$$

which proves the right inequality. □

5. Let O denote the circumcentre of an acute-angled triangle ABC. A circle Γ passing through vertex A intersects segments AB and AC at points P and Q such that $\angle BOP = \angle ABC$ and $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to Γ.

Solution. Let the circumcircle of triangle OBP intersect side BC at the points R and B and let $\angle A$, $\angle B$ and $\angle C$ denote the angles at vertices A, B and C, respectively.

Now note that since $\angle BOP = \angle B$ and $\angle COQ = \angle C$, it follows that

$$\angle POQ = 360^\circ - \angle BOP - \angle COQ - \angle BOC = 360^\circ - (180 - \angle A) - 2\angle A = 180^\circ - \angle A.$$

This implies that $APOQ$ is a cyclic quadrilateral. Since $BPOR$ is cyclic,

$$\angle QOR = 360^\circ - \angle POQ - \angle POR = 360^\circ - (180^\circ - \angle A) - (180^\circ - \angle B) = 180^\circ - \angle C.$$

This implies that $CQOR$ is a cyclic quadrilateral. Since $APOQ$ and $BPOR$ are cyclic,

$$\angle QPR = \angle QPO + \angle OPR = \angle OAQ + \angle OBR = (90^\circ - \angle B) + (90^\circ - \angle A) = \angle C.$$

Since $CQOR$ is cyclic, $\angle QRC = \angle COQ = \angle C = \angle QPR$ which implies that the circumcircle of triangle PQR is tangent to BC. Further, since $\angle PRB = \angle BOP = \angle B$,

$$\angle PRQ = 180^\circ - \angle PRB - \angle QRC = 180^\circ - \angle B - \angle C = \angle A = \angle PAQ.$$

This implies that the circumcircle of PQR is the reflection of Γ in line PQ. By symmetry in line PQ, this implies that the reflection of BC in line PQ is tangent to Γ.