Solutions to the 2005 CMO
written March 30, 2005

1. Consider an equilateral triangle of side length $n$, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.

Solution

We shall show that $f(n) = (n - 1)!$.

Label the horizontal line segments in the triangle $l_1$, $l_2$, ... as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $l_1, l_2, \ldots, l_{n-1}$ exactly once. The diagonal lines in the triangle divide $l_k$ into $k$ unit line segments and the path must cross exactly one of these $k$ segments for each $k$. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n - 1$ line segments which are crossed. So as the path moves from the $k$th row to the $(k + 1)$st row, there are $k$ possible line segments where the path could cross $l_k$. Since there are $1 \cdot 2 \cdot 3 \cdots (n - 1) = (n - 1)!$ ways that the path could cross the $n - 1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n - 1)!$.


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[Diagram of the equilateral triangle with labeled line segments $l_1, l_2, l_3, l_4$, and a path from the top triangle to the middle triangle in the bottom row.]
2. Let \((a, b, c)\) be a Pythagorean triple, \(i.e.,\) a triplet of positive integers with \(a^2 + b^2 = c^2\).

a) Prove that \((c/a + c/b)^2 > 8\).

b) Prove that there does not exist any integer \(n\) for which we can find a Pythagorean triple \((a, b, c)\) satisfying \((c/a + c/b)^2 = n\).

a) Solution 1

Let \((a, b, c)\) be a Pythagorean triple. View \(a, b\) as lengths of the legs of a right angled triangle with hypotenuse of length \(c\); let \(\theta\) be the angle determined by the sides with lengths \(a\) and \(c\). Then

\[
\left(\frac{c}{a} + \frac{c}{b}\right)^2 = \left(\frac{1}{\cos \theta} + \frac{1}{\sin \theta}\right)^2 = \frac{\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta}{(\sin \theta \cos \theta)^2}
\]

\[
= 4 \left(\frac{1 + \sin \theta \cos \theta}{\sin^2 \theta \cos \theta}\right) = \frac{4}{\sin^2 2\theta} + \frac{4}{\sin 2\theta}
\]

Note that because \(0 < \theta < 90^\circ\), we have \(0 < \sin 2\theta \leq 1\), with equality only if \(\theta = 45^\circ\). But then \(a = b\) and we obtain \(\sqrt{2} = c/a\), contradicting \(a, c\) both being integers. Thus, \(0 < \sin 2\theta < 1\) which gives \((c/a + c/b)^2 > 8\).

Solution 2

Defining \(\theta\) as in Solution 1, we have \(c/a + c/b = \sec \theta + \csc \theta\). By the AM-GM inequality, we have \((\sec \theta + \csc \theta)/2 \geq \sqrt{\sec \theta \csc \theta}\). So

\[
c/a + c/b \geq \frac{2}{\sqrt{\sin \theta \cos \theta}} = \frac{2\sqrt{2}}{\sqrt{\sin 2\theta}} \geq 2\sqrt{2}.
\]

Since \(a, b, c\) are integers, we have \(c/a + c/b > 2\sqrt{2}\) which gives \((c/a + c/b)^2 > 8\).

Solution 3

By simplifying and using the AM-GM inequality,

\[
\left(\frac{c}{a} + \frac{c}{b}\right)^2 = c^2 \left(\frac{a + b}{ab}\right)^2 = \frac{(a^2 + b^2)(a + b)^2}{a^2b^2} \geq 2\frac{a^2b^2(2\sqrt{ab})^2}{a^2b^2} = 8,
\]

with equality only if \(a = b\). By using the same argument as in Solution 1, \(a\) cannot equal \(b\) and the inequality is strict.

Solution 4

\[
\left(\frac{c}{a} + \frac{c}{b}\right)^2 = \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{2c^2}{ab} = 1 + \frac{b^2}{a^2} + \frac{a^2}{b^2} + 1 + \frac{2(a^2 + b^2)}{ab}
\]

\[
= 2 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + 2 + \frac{2}{ab} \left((a - b)^2 + 2ab\right)
\]

\[
= 4 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + \frac{2(a - b)^2}{ab} + 4 \geq 8,
\]

with equality only if \(a = b\), which (as argued previously) cannot occur.
b) **Solution 1**

Since \( c/a + c/b \) is rational, \((c/a + c/b)^2\) can only be an integer if \( c/a + c/b \) is an integer. Suppose \( c/a + c/b = m \). We may assume that \( \gcd(a, b) = 1 \). (If not, divide the common factor from \( (a, b, c) \), leaving \( m \) unchanged.)

Since \( c(a+b) = mab \) and \( \gcd(a, a+b) = 1 \), \( a \) must divide \( c \), say \( c = ak \). This gives \( a^2 + b^2 = a^2k^2 \) which implies \( b^2 = (k^2 - 1)a^2 \). But then \( a \) divides \( b \) contradicting the fact that \( \gcd(a, b) = 1 \). Therefore \((c/a + c/b)^2\) is not equal to any integer \( n \).

**Solution 2**

We begin as in Solution 1, supposing that \( c/a + c/b = m \) with \( \gcd(a, b) = 1 \). Hence \( a \) and \( b \) are not both even. It is also the case that \( a \) and \( b \) are not both odd, for then \( c^2 = a^2 + b^2 \equiv 2 \pmod{4} \), and perfect squares are congruent to either 0 or 1 modulo 4. So one of \( a, b \) is odd and the other is even. Therefore \( c \) must be odd.

Now \( c/a + c/b = m \) implies \( c(a+b) = mab \), which cannot be true because \( c(a+b) \) is odd and \( mab \) is even.
3. Let $S$ be a set of $n \geq 3$ points in the interior of a circle.

a) Show that there are three distinct points $a, b, c \in S$ and three distinct points $A, B, C$ on the circle such that $a$ is (strictly) closer to $A$ than any other point in $S$, $b$ is closer to $B$ than any other point in $S$ and $c$ is closer to $C$ than any other point in $S$.

b) Show that for no value of $n$ can four such points in $S$ (and corresponding points on the circle) be guaranteed.

Solution 1

a) Let $H$ be the smallest convex set of points in the plane which contains $S$.† Take 3 points $a, b, c \in S$ which lie on the boundary of $H$. (There must always be at least 3 (but not necessarily 4) such points.)

Since $a$ lies on the boundary of the convex region $H$, we can construct a chord $L$ such that no two points of $H$ lie on opposite sides of $L$. Of the two points where the perpendicular to $L$ at $a$ meets the circle, choose one which is on a side of $L$ not containing any points of $H$ and call this point $A$. Certainly $A$ is closer to $a$ than to any other point on $L$ or on the other side of $L$. Hence $A$ is closer to $a$ than to any other point of $S$. We can find the required points $B$ and $C$ in an analogous way and the proof is complete.

[Note that this argument still holds if all the points of $S$ lie on a line.]

\[ \text{(a)} \]

b) Let $PQR$ be an equilateral triangle inscribed in the circle and let $a, b, c$ be mid-points of the three sides of $\triangle PQR$. If $r$ is the radius of the circle, then every point on the circle is within $(\sqrt{3}/2)r$ of one of $a, b$ or $c$. (See figure (b) above.)

Now $\sqrt{3}/2 < 9/10$, so if $S$ consists of $a, b, c$ and a cluster of points within $r/10$ of the centre of the circle, then we cannot select 4 points from $S$ (and corresponding points on the circle) having the desired property.

†By the way, $H$ is called the convex hull of $S$. If the points of $S$ lie on a line, then $H$ will be the shortest line segment containing the points of $S$. Otherwise, $H$ is a polygon whose vertices are all elements of $S$ and such that all other points in $S$ lie inside or on this polygon.
Solution 2

a) If all the points of $S$ lie on a line $L$, then choose any 3 of them to be $a, b, c$. Let $A$ be a point on the circle which meets the perpendicular to $L$ at $a$. Clearly $A$ is closer to $a$ than to any other point on $L$, and hence closer than other other point in $S$. We find $B$ and $C$ in an analogous way.

Otherwise, choose $a, b, c$ from $S$ so that the triangle formed by these points has maximal area. Construct the altitude from the side $bc$ to the point $a$ and extend this line until it meets the circle at $A$. We claim that $A$ is closer to $a$ than to any other point in $S$.

Suppose not. Let $x$ be a point in $S$ for which the distance from $A$ to $x$ is less than the distance from $A$ to $a$. Then the perpendicular distance from $x$ to the line $bc$ must be greater than the perpendicular distance from $a$ to the line $bc$. But then the triangle formed by the points $x, b, c$ has greater area than the triangle formed by $a, b, c$, contradicting the original choice of these 3 points. Therefore $A$ is closer to $a$ than to any other point in $S$.

The points $B$ and $C$ are found by constructing similar altitudes through $b$ and $c$, respectively.

b) See Solution 1.

**Solution 1**

Since similar triangles give the same value of $KP/R^3$, we can fix $R = 1$ and maximize $KP$ over all triangles inscribed in the unit circle. Fix points $A$ and $B$ on the unit circle. The locus of points $C$ with a given perimeter $P$ is an ellipse that meets the circle in at most four points. The area $K$ is maximized (for a fixed $P$) when $C$ is chosen on the perpendicular bisector of $AB$, so we get a maximum value for $KP$ if $C$ is where the perpendicular bisector of $AB$ meets the circle. Thus the maximum value of $KP$ for a given $AB$ occurs when $ABC$ is an isosceles triangle. Repeating this argument with $BC$ fixed, we have that the maximum occurs when $ABC$ is an equilateral triangle.

Consider an equilateral triangle with side length $a$. It has $P = 3a$. It has height equal to $a\sqrt{3}/2$ giving $K = a^2\sqrt{3}/4$. From the extended law of sines, $2R = a/\sin(60)$ giving $R = a/\sqrt{3}$. Therefore the maximum value we seek is

$$KP/R^3 = \left(\frac{a^2\sqrt{3}}{4}\right)(3a)\left(\frac{\sqrt{3}}{a}\right)^3 = \frac{27}{4}.$$

**Solution 2**

From the extended law of sines, the lengths of the sides of the triangle are $2R\sin A$, $2R\sin B$ and $2R\sin C$. So

$$P = 2R(\sin A + \sin B + \sin C) \quad \text{and} \quad K = \frac{1}{2}(2R\sin A)(2R\sin B)(\sin C),$$

giving

$$\frac{KP}{R^3} = 4\sin A \sin B \sin C (\sin A + \sin B + \sin C).$$

We wish to find the maximum value of this expression over all $A + B + C = 180^\circ$. Using well-known identities for sums and products of sine functions, we can write

$$\frac{KP}{R^3} = 4\sin A \left(\frac{\cos(B - C)}{2} - \frac{\cos(B + C)}{2}\right) \left(\sin A + 2\sin \left(\frac{B + C}{2}\right) \cos \left(\frac{B - C}{2}\right)\right).$$

If we first consider $A$ to be fixed, then $B + C$ is fixed also and this expression takes its maximum value when $\cos(B - C)$ and $\cos \left(\frac{B - C}{2}\right)$ equal 1; i.e. when $B = C$. In a similar way, one can show that for any fixed value of $B$, $KP/R^3$ is maximized when $A = C$. Therefore the maximum value of $KP/R^3$ occurs when $A = B = C = 60^\circ$, and it is now an easy task to substitute this into the above expression to obtain the maximum value of $27/4$. 
Solution 3
As in Solution 2, we obtain
\[
\frac{KP}{R^3} = 4 \sin A \sin B \sin C (\sin A + \sin B + \sin C).
\]
From the AM-GM inequality, we have
\[
\sin A \sin B \sin C \leq \left( \frac{\sin A + \sin B + \sin C}{3} \right)^3,
\]
giving
\[
\frac{KP}{R^3} \leq \frac{4}{27} (\sin A + \sin B + \sin C)^4,
\]
with equality when \( \sin A = \sin B = \sin C \). Since the sine function is concave on the interval from 0 to \( \pi \), Jensen’s inequality gives
\[
\frac{\sin A + \sin B + \sin C}{3} \leq \sin \left( \frac{A + B + C}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.
\]
Since equality occurs here when \( \sin A = \sin B = \sin C \) also, we can conclude that the maximum value of \( KP/R^3 \) is \( \frac{4}{27} \left( \frac{3\sqrt{3}}{2} \right)^4 = 27/4 \).
5. Let’s say that an ordered triple of positive integers \((a, b, c)\) is \(n\)-powerful if \(a \leq b \leq c\), \(\gcd(a, b, c) = 1\), and \(a^n + b^n + c^n\) is divisible by \(a + b + c\). For example, \((1, 2, 2)\) is 5-powerful.

a) Determine all ordered triples (if any) which are \(n\)-powerful for all \(n \geq 1\).

b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

[Note that \(\gcd(a, b, c)\) is the greatest common divisor of \(a\), \(b\) and \(c\).]

**Solution 1**

Let \(T_n = a^n + b^n + c^n\) and consider the polynomial

\[
P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc.
\]

Since \(P(a) = 0\), we get \(a^3 = (a + b + c)a^2 - (ab + ac + bc)a + abc\) and multiplying both sides by \(a^{n-3}\) we obtain \(a^n = (a + b + c)a^{n-1} - (ab + ac + bc)a^{n-2} + (abc)a^{n-3}\). Applying the same reasoning, we can obtain similar expressions for \(b^n\) and \(c^n\) and adding the three identities we get that \(T_n\) satisfies the following 3-term recurrence:

\[
T_n = (a + b + c)T_{n-1} - (ab + ac + bc)T_{n-2} + (abc)T_{n-3}, \quad \text{for all } n \geq 3.
\]

From this we see that if \(T_{n-2}\) and \(T_{n-3}\) are divisible by \(a + b + c\), then so is \(T_n\). This immediately resolves part (b)—there are no ordered triples which are 2004-powerful and 2005-powerful, but not 2007-powerful—and reduces the number of cases to be considered in part (a): since all triples are 1-powerful, the recurrence implies that any ordered triple which is both 2-powerful and 3-powerful is \(n\)-powerful for all \(n \geq 1\).

Putting \(n = 3\) in the recurrence, we have

\[
a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2) - (ab + ac + bc)(a + b + c) + 3abc
\]

which implies that \((a, b, c)\) is 3-powerful if and only if \(3abc\) is divisible by \(a + b + c\). Since

\[
a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc),
\]

\((a, b, c)\) is 2-powerful if and only if \(2(ab + ac + bc)\) is divisible by \(a + b + c\).

Suppose a prime \(p \geq 5\) divides \(a + b + c\). Then \(p\) divides \(abc\). Since \(\gcd(a, b, c) = 1\), \(p\) divides exactly one of \(a, b\) or \(c\); but then \(p\) doesn’t divide \(2(ab + ac + bc)\).

Suppose \(3^2\) divides \(a + b + c\). Then \(3\) divides \(abc\), implying \(3\) divides exactly one of \(a, b\) or \(c\). But then \(3\) doesn’t divide \(2(ab + ac + bc)\).

Suppose \(2^2\) divides \(a + b + c\). Then \(4\) divides \(abc\). Since \(\gcd(a, b, c) = 1\), at most one of \(a, b\) or \(c\) is even, implying one of \(a, b, c\) is divisible by \(4\) and the others are odd. But then \(ab + ac + bc\) is odd and \(4\) doesn’t divide \(2(ab + ac + bc)\).

So if \((a, b, c)\) is 2- and 3-powerful, then \(a + b + c\) is not divisible by 4 or 9 or any prime greater than 3. Since \(a + b + c\) is at least 3, \(a + b + c\) is either 3 or 6. It is now a simple matter to check the possibilities and conclude that the only triples which are \(n\)-powerful for all \(n \geq 1\) are \((1, 1, 1)\) and \((1, 1, 4)\).
Solution 2

Let $p$ be a prime. By Fermat’s Little Theorem,

$$a^{p-1} \equiv \begin{cases} 1 \text{ (mod } p) , & \text{ if } p \text{ doesn’t divide } a; \\ 0 \text{ (mod } p) , & \text{ if } p \text{ divides } a. \end{cases}$$

Since $\gcd(a, b, c) = 1$, we have that $a^{p-1} + b^{p-1} + c^{p-1} \equiv 1, 2$ or $3$ (mod $p$). Therefore if $p$ is a prime divisor of $a^{p-1} + b^{p-1} + c^{p-1}$, then $p$ equals $2$ or $3$. So if $(a, b, c)$ is $n$-powerful for all $n \geq 1$, then the only primes which can divide $a + b + c$ are $2$ or $3$.

We can proceed in a similar fashion to show that $a + b + c$ is not divisible by $4$ or $9$. Since

$$a^2 \equiv \begin{cases} 0 \text{ (mod } 4) , & \text{ if } p \text{ is even; } \\ 1 \text{ (mod } 4) , & \text{ if } p \text{ is odd} \end{cases}$$

and $a, b, c$ aren’t all even, we have that $a^2 + b^2 + c^2 \equiv 1, 2$ or $3$ (mod $4$).

By expanding $(3k)^3$, $(3k + 1)^3$ and $(3k + 2)^3$, we find that $a^3$ is congruent to $0$, $1$ or $-1$ modulo $9$. Hence

$$a^6 \equiv \begin{cases} 0 \text{ (mod } 9) , & \text{ if } 3 \text{ divides } a; \\ 1 \text{ (mod } 9) , & \text{ if } 3 \text{ doesn’t divide } a. \end{cases}$$

Since $a, b, c$ aren’t all divisible by $3$, we have that $a^6 + b^6 + c^6 \equiv 1, 2$ or $3$ (mod $9$).

So $a^2 + b^2 + c^2$ is not divisible by $4$ and $a^6 + b^6 + c^6$ is not divisible by $9$. Thus if $(a, b, c)$ is $n$-powerful for all $n \geq 1$, then $a + b + c$ is not divisible by $4$ or $9$. Therefore $a + b + c$ is either $3$ or $6$ and checking all possibilities, we conclude that the only triples which are $n$-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

See Solution 1 for the (b) part.