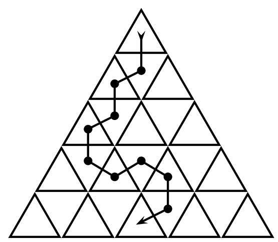
Solutions to the 2005 CMO

written March 30, 2005

1. Consider an equilateral triangle of side length n, which is divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for $n = 5$. Determine the value of $f(2005)$.

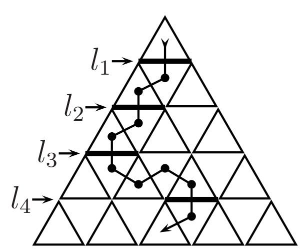


Solution

We shall show that $f(n) = (n-1)!$.

Label the horizontal line segments in the triangle l_1, l_2, ... as in the diagram below. Since the path goes from the top triangle to a triangle in the bottom row and never travels up, the path must cross each of $l_1, l_2, \ldots, l_{n-1}$ exactly once. The diagonal lines in the triangle divide l_k into k unit line segments and the path must cross exactly one of these k segments for each k. (In the diagram below, these line segments have been highlighted.) The path is completely determined by the set of $n-1$ line segments which are crossed. So as the path moves from the kth row to the $(k+1)$st row, there are k possible line segments where the path could cross l_k. Since there are $1 \cdot 2 \cdot 3 \cdots (n-1) = (n-1)!$ ways that the path could cross the $n-1$ horizontal lines, and each one corresponds to a unique path, we get $f(n) = (n-1)!$.

Therefore $f(2005) = (2004)!$.



2. Let (a, b, c) be a Pythagorean triple, *i.e.*, a triplet of positive integers with $a^2 + b^2 = c^2$.

   a) Prove that $(c/a + c/b)^2 > 8$.

   b) Prove that there does not exist any integer n for which we can find a Pythagorean triple (a, b, c) satisfying $(c/a + c/b)^2 = n$.

a) **Solution 1**

Let (a, b, c) be a Pythagorean triple. View a, b as lengths of the legs of a right angled triangle with hypotenuse of length c; let θ be the angle determined by the sides with lengths a and c. Then

$$\begin{aligned}\left(\frac{c}{a} + \frac{c}{b}\right)^2 &= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)^2 = \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2} \\ &= 4\left(\frac{1 + \sin 2\theta}{\sin^2 2\theta}\right) = \frac{4}{\sin^2 2\theta} + \frac{4}{\sin 2\theta}\end{aligned}$$

Note that because $0 < \theta < 90°$, we have $0 < \sin 2\theta \leq 1$, with equality only if $\theta = 45°$. But then $a = b$ and we obtain $\sqrt{2} = c/a$, contradicting a, c both being integers. Thus, $0 < \sin 2\theta < 1$ which gives $(c/a + c/b)^2 > 8$.

Solution 2

Defining θ as in Solution 1, we have $c/a + c/b = \sec\theta + \csc\theta$. By the AM-GM inequality, we have $(\sec\theta + \csc\theta)/2 \geq \sqrt{\sec\theta\csc\theta}$. So

$$c/a + c/b \geq \frac{2}{\sqrt{\sin\theta\cos\theta}} = \frac{2\sqrt{2}}{\sqrt{\sin 2\theta}} \geq 2\sqrt{2}.$$

Since a, b, c are integers, we have $c/a + c/b > 2\sqrt{2}$ which gives $(c/a + c/b)^2 > 8$.

Solution 3

By simplifying and using the AM-GM inequality,

$$\left(\frac{c}{a} + \frac{c}{b}\right)^2 = c^2\left(\frac{a+b}{ab}\right)^2 = \frac{(a^2+b^2)(a+b)^2}{a^2b^2} \geq \frac{2\sqrt{a^2b^2}\,(2\sqrt{ab})^2}{a^2b^2} = 8,$$

with equality only if $a = b$. By using the same argument as in Solution 1, a cannot equal b and the inequality is strict.

Solution 4

$$\begin{aligned}\left(\frac{c}{a} + \frac{c}{b}\right)^2 &= \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{2c^2}{ab} = 1 + \frac{b^2}{a^2} + \frac{a^2}{b^2} + 1 + \frac{2(a^2+b^2)}{ab} \\ &= 2 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + 2 + \frac{2}{ab}\Big((a-b)^2 + 2ab\Big) \\ &= 4 + \left(\frac{a}{b} - \frac{b}{a}\right)^2 + \frac{2(a-b)^2}{ab} + 4 \geq 8,\end{aligned}$$

with equality only if $a = b$, which (as argued previously) cannot occur.

b) **Solution 1**

Since $c/a + c/b$ is rational, $(c/a + c/b)^2$ can only be an integer if $c/a + c/b$ is an integer. Suppose $c/a + c/b = m$. We may assume that $\gcd(a, b) = 1$. (If not, divide the common factor from (a, b, c), leaving m unchanged.)

Since $c(a+b) = mab$ and $\gcd(a, a+b) = 1$, a must divide c, say $c = ak$. This gives $a^2 + b^2 = a^2k^2$ which implies $b^2 = (k^2 - 1)a^2$. But then a divides b contradicting the fact that $\gcd(a, b) = 1$. Therefore $(c/a + c/b)^2$ is not equal to any integer n.

Solution 2

We begin as in Solution 1, supposing that $c/a + c/b = m$ with $\gcd(a, b) = 1$. Hence a and b are not both even. It is also the case that a and b are not both odd, for then $c^2 = a^2 + b^2 \equiv 2 \pmod 4$, and perfect squares are congruent to either 0 or 1 modulo 4. So one of a, b is odd and the other is even. Therefore c must be odd.

Now $c/a + c/b = m$ implies $c(a+b) = mab$, which cannot be true because $c(a+b)$ is odd and mab is even.

3. Let S be a set of $n \geq 3$ points in the interior of a circle.

a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is (strictly) closer to A than any other point in S, b is closer to B than any other point in S and c is closer to C than any other point in S.

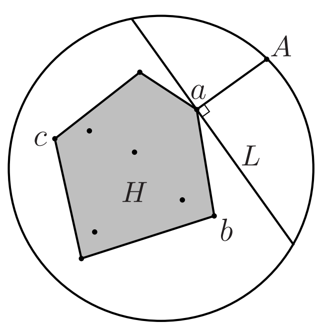
b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

Solution 1

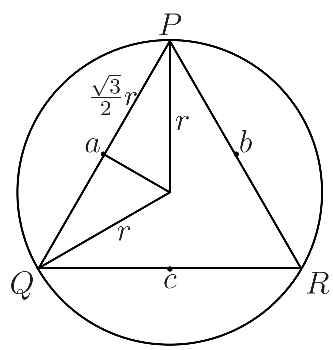
a) Let H be the smallest convex set of points in the plane which contains S.† Take 3 points $a, b, c \in S$ which lie on the boundary of H. (There must always be at least 3 (but not necessarily 4) such points.)

Since a lies on the boundary of the convex region H, we can construct a chord L such that no two points of H lie on opposite sides of L. Of the two points where the perpendicular to L at a meets the circle, choose one which is on a side of L not containing any points of H and call this point A. Certainly A is closer to a than to any other point on L or on the other side of L. Hence A is closer to a than to any other point of S. We can find the required points B and C in an analogous way and the proof is complete.

[Note that this argument still holds if all the points of S lie on a line.]



(a)



(b)

b) Let PQR be an equilateral triangle inscribed in the circle and let a, b, c be mid-points of the three sides of $\triangle PQR$. If r is the radius of the circle, then every point on the circle is within $(\sqrt{3}/2)r$ of one of a, b or c. (See figure (b) above.)

Now $\sqrt{3}/2 < 9/10$, so if S consists of a, b, c and a cluster of points within $r/10$ of the centre of the circle, then we cannot select 4 points from S (and corresponding points on the circle) having the desired property.

†By the way, H is called the convex hull of S. If the points of S lie on a line, then H will be the shortest line segment containing the points of S. Otherwise, H is a polygon whose vertices are all elements of S and such that all other points in S lie inside or on this polygon.

Solution 2

a) If all the points of S lie on a line L, then choose any 3 of them to be a, b, c. Let A be a point on the circle which meets the perpendicular to L at a. Clearly A is closer to a than to any other point on L, and hence closer than other other point in S. We find B and C in an analogous way.

   Otherwise, choose a, b, c from S so that the triangle formed by these points has maximal area. Construct the altitude from the side bc to the point a and extend this line until it meets the circle at A. We claim that A is closer to a than to any other point in S.

   Suppose not. Let x be a point in S for which the distance from A to x is less than the distance from A to a. Then the perpendicular distance from x to the line bc must be greater than the perpendicular distance from a to the line bc. But then the triangle formed by the points x, b, c has greater area than the triangle formed by a, b, c, contradicting the original choice of these 3 points. Therefore A is closer to a than to any other point in S.

   The points B and C are found by constructing similar altitudes through b and c, respectively.

b) See Solution 1.

4. Let ABC be a triangle with circumradius R, perimeter P and area K. Determine the maximum value of KP/R^3.

Solution 1

Since similar triangles give the same value of KP/R^3, we can fix $R = 1$ and maximize KP over all triangles inscribed in the unit circle. Fix points A and B on the unit circle. The locus of points C with a given perimeter P is an ellipse that meets the circle in at most four points. The area K is maximized (for a fixed P) when C is chosen on the perpendicular bisector of AB, so we get a maximum value for KP if C is where the perpendicular bisector of AB meets the circle. Thus the maximum value of KP for a given AB occurs when ABC is an isosceles triangle. Repeating this argument with BC fixed, we have that the maximum occurs when ABC is an equilateral triangle.

Consider an equilateral triangle with side length a. It has $P = 3a$. It has height equal to $a\sqrt{3}/2$ giving $K = a^2\sqrt{3}/4$. ¿From the extended law of sines, $2R = a/\sin(60)$ giving $R = a/\sqrt{3}$. Therefore the maximum value we seek is

$$KP/R^3 = \left(\frac{a^2\sqrt{3}}{4}\right)(3a)\left(\frac{\sqrt{3}}{a}\right)^3 = \frac{27}{4}.$$

Solution 2

From the extended law of sines, the lengths of the sides of the triangle are $2R\sin A$, $2R\sin B$ and $2R\sin C$. So

$$P = 2R(\sin A + \sin B + \sin C) \text{ and } K = \frac{1}{2}(2R\sin A)(2R\sin B)(\sin C),$$

giving

$$\frac{KP}{R^3} = 4\sin A\sin B\sin C(\sin A + \sin B + \sin C).$$

We wish to find the maximum value of this expression over all $A + B + C = 180°$. Using well-known identities for sums and products of sine functions, we can write

$$\frac{KP}{R^3} = 4\sin A\left(\frac{\cos(B-C)}{2} - \frac{\cos(B+C)}{2}\right)\left(\sin A + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right).$$

If we first consider A to be fixed, then $B + C$ is fixed also and this expression takes its maximum value when $\cos(B - C)$ and $\cos\left(\frac{B-C}{2}\right)$ equal 1; *i.e.* when $B = C$. In a similar way, one can show that for any fixed value of B, KP/R^3 is maximized when $A = C$. Therefore the maximum value of KP/R^3 occurs when $A = B = C = 60°$, and it is now an easy task to substitute this into the above expression to obtain the maximum value of $27/4$.

Solution 3

As in Solution 2, we obtain

$$\frac{KP}{R^3} = 4\sin A \sin B \sin C(\sin A + \sin B + \sin C).$$

From the AM-GM inequality, we have

$$\sin A \sin B \sin C \leq \left(\frac{\sin A + \sin B + \sin C}{3}\right)^3,$$

giving

$$\frac{KP}{R^3} \leq \frac{4}{27}(\sin A + \sin B + \sin C)^4,$$

with equality when $\sin A = \sin B = \sin C$. Since the sine function is concave on the interval from 0 to π, Jensen's inequality gives

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A + B + C}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Since equality occurs here when $\sin A = \sin B = \sin C$ also, we can conclude that the maximum value of KP/R^3 is $\frac{4}{27}\left(\frac{3\sqrt{3}}{2}\right)^4 = 27/4$.

5. Let's say that an ordered triple of positive integers (a, b, c) is *n-powerful* if $a \leq b \leq c$, $\gcd(a, b, c) = 1$, and $a^n + b^n + c^n$ is divisible by $a + b + c$. For example, $(1, 2, 2)$ is 5-powerful.

   a) Determine all ordered triples (if any) which are n-powerful for all $n \geq 1$.

   b) Determine all ordered triples (if any) which are 2004-powerful and 2005-powerful, but not 2007-powerful.

   [Note that $\gcd(a, b, c)$ is the greatest common divisor of a, b and c.]

Solution 1

Let $T_n = a^n + b^n + c^n$ and consider the polynomial

$$P(x) = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc.$$

Since $P(a) = 0$, we get $a^3 = (a+b+c)a^2 - (ab+ac+bc)a + abc$ and multiplying both sides by a^{n-3} we obtain $a^n = (a+b+c)a^{n-1} - (ab+ac+bc)a^{n-2} + (abc)a^{n-3}$. Applying the same reasoning, we can obtain similar expressions for b^n and c^n and adding the three identities we get that T_n satisfies the following 3-term recurrence:

$$T_n = (a + b + c)T_{n-1} - (ab + ac + bc)T_{n-2} + (abc)T_{n-3}, \text{ for all } n \geq 3.$$

¿From this we see that if T_{n-2} and T_{n-3} are divisible by $a + b + c$, then so is T_n. This immediately resolves part (b)—there are no ordered triples which are 2004-powerful and 2005-powerful, but not 2007-powerful—and reduces the number of cases to be considered in part (a): since all triples are 1-powerful, the recurrence implies that any ordered triple which is both 2-powerful and 3-powerful is n-powerful for all $n \geq 1$.

Putting $n = 3$ in the recurrence, we have

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2) - (ab + ac + bc)(a + b + c) + 3abc$$

which implies that (a, b, c) is 3-powerful if and only if $3abc$ is divisible by $a + b + c$. Since

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc),$$

(a, b, c) is 2-powerful if and only if $2(ab + ac + bc)$ is divisible by $a + b + c$.

Suppose a prime $p \geq 5$ divides $a + b + c$. Then p divides abc. Since $\gcd(a, b, c) = 1$, p divides exactly one of a, b or c; but then p doesn't divide $2(ab + ac + bc)$.

Suppose 3^2 divides $a + b + c$. Then 3 divides abc, implying 3 divides exactly one of a, b or c. But then 3 doesn't divide $2(ab + ac + bc)$.

Suppose 2^2 divides $a + b + c$. Then 4 divides abc. Since $\gcd(a, b, c) = 1$, at most one of a, b or c is even, implying one of a, b, c is divisible by 4 and the others are odd. But then $ab + ac + bc$ is odd and 4 doesn't divide $2(ab + ac + bc)$.

So if (a, b, c) is 2- and 3-powerful, then $a + b + c$ is not divisible by 4 or 9 or any prime greater than 3. Since $a + b + c$ is at least 3, $a + b + c$ is either 3 or 6. It is now a simple matter to check the possibilities and conclude that the only triples which are n-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

Solution 2

Let p be a prime. By Fermat's Little Theorem,

$$a^{p-1} \equiv \begin{cases} 1 \pmod{p}, & \text{if } p \text{ doesn't divide } a; \\ 0 \pmod{p}, & \text{if } p \text{ divides } a. \end{cases}$$

Since $\gcd(a, b, c) = 1$, we have that $a^{p-1} + b^{p-1} + c^{p-1} \equiv 1, 2$ or $3 \pmod{p}$. Therefore if p is a prime divisor of $a^{p-1} + b^{p-1} + c^{p-1}$, then p equals 2 or 3. So if (a, b, c) is n-powerful for all $n \geq 1$, then the only primes which can divide $a + b + c$ are 2 or 3.

We can proceed in a similar fashion to show that $a + b + c$ is not divisible by 4 or 9.

Since

$$a^2 \equiv \begin{cases} 0 \pmod{4}, & \text{if } p \text{ is even;} \\ 1 \pmod{4}, & \text{if } p \text{ is odd} \end{cases}$$

and a, b, c aren't all even, we have that $a^2 + b^2 + c^2 \equiv 1, 2$ or $3 \pmod{4}$.

By expanding $(3k)^3$, $(3k + 1)^3$ and $(3k + 2)^3$, we find that a^3 is congruent to 0, 1 or -1 modulo 9. Hence

$$a^6 \equiv \begin{cases} 0 \pmod{9}, & \text{if } 3 \text{ divides } a; \\ 1 \pmod{9}, & \text{if } 3 \text{ doesn't divide } a. \end{cases}$$

Since a, b, c aren't all divisible by 3, we have that $a^6 + b^6 + c^6 \equiv 1, 2$ or $3 \pmod{9}$.

So $a^2 + b^2 + c^2$ is not divisible by 4 and $a^6 + b^6 + c^6$ is not divisible by 9. Thus if (a, b, c) is n-powerful for all $n \geq 1$, then $a + b + c$ is not divisible by 4 or 9. Therefore $a + b + c$ is either 3 or 6 and checking all possibilities, we conclude that the only triples which are n-powerful for all $n \geq 1$ are $(1, 1, 1)$ and $(1, 1, 4)$.

See Solution 1 for the (b) part.