Solutions to the 2003 CMO

written March 26, 2003

1. Consider a standard twelve-hour clock whose hour and minute hands move continuously. Let m be an integer, with $1 \le m \le 720$. At precisely m minutes after 12:00, the angle made by the hour hand and minute hand is exactly 1°. Determine all possible values of m.

   Solution

   The minute hand makes a full revolution of 360° every 60 minutes, so after m minutes it has swept through $\frac{360}{60}m = 6m$ degrees. The hour hand makes a full revolution every 12 hours (720 minutes), so after m minutes it has swept through $\frac{360}{720}m = m/2$ degrees. Since both hands started in the same position at 12:00, the angle between the two hands will be 1° if $6m - m/2 = \pm 1 + 360k$ for some integer k. Solving this equation we get

   $$m = \frac{720k \pm 2}{11} = 65k + \frac{5k \pm 2}{11}.$$

   Since $1 \le m \le 720$, we have $1 \le k \le 11$. Since m is an integer, $5k \pm 2$ must be divisible by 11, say $5k \pm 2 = 11q$. Then

   $$5k = 11q \pm 2 \quad \Rightarrow \quad k = 2q + \frac{q \pm 2}{5}.$$

   If is now clear that only $q = 2$ and $q = 3$ satisfy all the conditions. Thus $k = 4$ or $k = 7$ and substituting these values into the expression for m we find that the only possible values of m are 262 and 458.

2. Find the last three digits of the number $2003^{2002^{2001}}$.

Solution

We must find the remainder when $2003^{2002^{2001}}$ is divided by 1000, which will be the same as the remainder when $3^{2002^{2001}}$ is divided by 1000, since $2003 \equiv 3 \pmod{1000}$. To do this we will first find a positive integer n such that $3^n \equiv 1 \pmod{1000}$ and then try to express 2002^{2001} in the form $nk + r$, so that

$$2003^{2002^{2001}} \equiv 3^{nk+r} \equiv (3^n)^k \cdot 3^r \equiv 1^k \cdot 3^r \equiv 3^r \pmod{1000}.$$

Since $3^2 = 10 - 1$, we can evaluate 3^{2m} using the binomial theorem:

$$3^{2m} = (10-1)^m = (-1)^m + 10m(-1)^{m-1} + 100\frac{m(m-1)}{2}(-1)^{m-2} + \cdots + 10^m.$$

After the first 3 terms of this expansion, all remaining terms are divisible by 1000, so letting $m = 2q$, we have that

$$3^{4q} \equiv 1 - 20q + 100q(2q-1) \pmod{1000}. \tag{1}$$

Using this, we can check that $3^{100} \equiv 1 \pmod{1000}$ and now we wish to find the remainder when 2002^{2001} is divided by 100.

Now $2002^{2001} \equiv 2^{2001} \pmod{100} \equiv 4 \cdot 2^{1999} \pmod{4 \cdot 25}$, so we'll investigate powers of 2 modulo 25. Noting that $2^{10} = 1024 \equiv -1 \pmod{25}$, we have

$$2^{1999} = (2^{10})^{199} \cdot 2^9 \equiv (-1)^{199} \cdot 512 \equiv -12 \equiv 13 \pmod{25}.$$

Thus $2^{2001} \equiv 4 \cdot 13 = 52 \pmod{100}$. Therefore 2002^{2001} can be written in the form $100k + 52$ for some integer k, so

$$2003^{2002^{2001}} \equiv 3^{52} \pmod{1000} \equiv 1 - 20 \cdot 13 + 1300 \cdot 25 \equiv 241 \pmod{1000}$$

using equation (1). So the last 3 digits of $2003^{2002^{2001}}$ are 241.

3. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$

$$x^2 + y^2 + z^2 = xyz.$$

Solution 1

Let $f(x, y, z) = (x^3 - x) + (y^3 - y) + (z^3 - z)$. The first equation above is equivalent to $f(x, y, z) = 0$. If $x, y, z \geq 1$, then $f(x, y, z) \geq 0$ with equality only if $x = y = z = 1$. But if $x = y = z = 1$, then the second equation is not satisfied. So in any solution to the system of equations, at least one of the variables is less than 1. Without loss of generality, suppose that $x < 1$. Then

$$x^2 + y^2 + z^2 > y^2 + z^2 \geq 2yz > yz > xyz.$$

Therefore the system has no real positive solutions.

Solution 2

We will show that the system has no real positive solution. Assume otherwise.

The second equation can be written $x^2 - (yz)x + (y^2 + z^2)$. Since this quadratic in x has a real solution by hypothesis, its discrimant is nonnegative. Hence

$$y^2z^2 - 4y^2 - 4z^2 \geq 0.$$

Dividing through by $4y^2z^2$ yields

$$\frac{1}{4} \geq \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{1}{y^2}.$$

Hence $y^2 \geq 4$ and so $y \geq 2$, y being positive. A similar argument yields $x, y, z \geq 2$. But the first equation can be written as

$$x(x^2 - 1) + y(y^2 - 1) + z(z^2 - 1) = 0,$$

contradicting $x, y, z \geq 2$. Hence, a real positive solution cannot exist.

Solution 3

Applying the arithmetic-geometric mean inequality and the Power Mean Inequalities to x, y, z we have

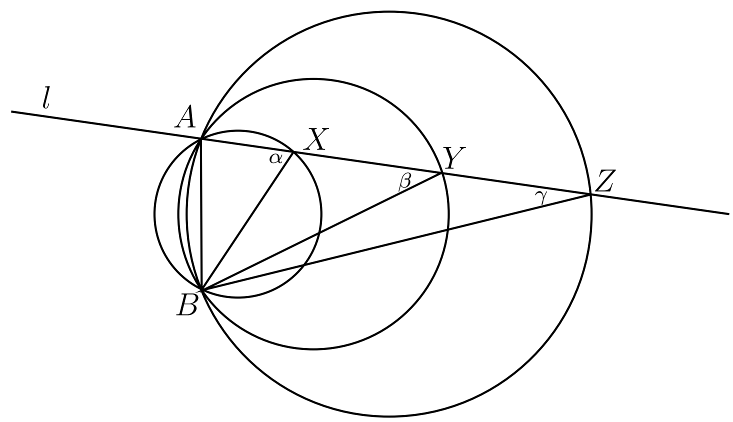
$$\sqrt[3]{xyz} \leq \frac{x+y+z}{3} \leq \sqrt{\frac{x^2+y^2+z^2}{3}} \leq \sqrt[3]{\frac{x^3+y^3+z^3}{3}}.$$

Letting $S = x+y+z = x^3+y^3+z^3$ and $P = xyz = x^2+y^2+z^2$, this inequality can be written

$$\sqrt[3]{P} \leq \frac{S}{3} \leq \sqrt{\frac{P}{3}} \leq \sqrt[3]{\frac{S}{3}}.$$

Now $\sqrt[3]{P} \leq \sqrt{\frac{P}{3}}$ implies $P^2 \leq P^3/27$, so $P \geq 27$. Also $\frac{S}{3} \leq \sqrt[3]{\frac{S}{3}}$ implies $S^3/27 \leq S/3$, so $S \leq 3$. But then $\sqrt[3]{P} \geq 3$ and $\sqrt[3]{\frac{S}{3}} \leq 1$ which is inconsistent with $\sqrt[3]{P} \leq \sqrt[3]{\frac{S}{3}}$. Therefore the system cannot have a real positive solution.

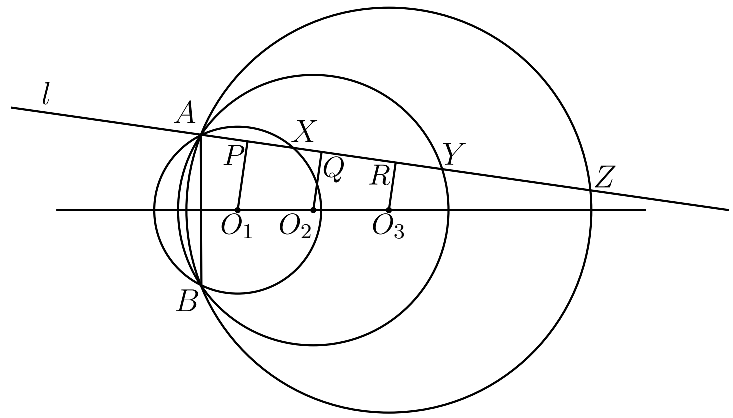
4. Prove that when three circles share the same chord AB, every line through A different from AB determines the same ratio $XY : YZ$, where X is an arbitrary point different from B on the first circle while Y and Z are the points where AX intersects the other two circles (labelled so that Y is between X and Z).



Solution 1

Let l be a line through A different from AB and join B to A, X, Y and Z as in the above diagram. No matter how l is chosen, the angles AXB, AYB and AZB always subtend the chord AB. For this reason the angles in the triangles BXY and BXZ are the same for all such l. Thus the ratio $XY : YZ$ remains constant by similar triangles.

Note that this is true no matter how X, Y and Z lie in relation to A. Suppose X, Y and Z all lie on the same side of A (as in the diagram) and that $\measuredangle AXB = \alpha$, $\measuredangle AYB = \beta$ and $\measuredangle AZB = \gamma$. Then $\measuredangle BXY = 180^\circ - \alpha$, $\measuredangle BYX = \beta$, $\measuredangle BYZ = 180^\circ - \beta$ and $\measuredangle BZY = \gamma$. Now suppose l is chosen so that X is now on the opposite side of A from Y and Z. Now since X is on the other side of the chord AB, $\measuredangle AXB = 180^\circ - \alpha$, but it is still the case that $\measuredangle BXY = 180^\circ - \alpha$ and all other angles in the two pertinent triangles remain unchanged. If l is chosen so that X is identical with A, then l is tangent to the first circle and it is still the case that $\measuredangle BXY = 180^\circ - \alpha$. All other cases can be checked in a similar manner.



Solution 2

Let m be the perpendicular bisector of AB and let O_1, O_2, O_3 be the centres of the three circles. Since AB is a chord common to all three circles, O_1, O_2, O_3 all lie on m. Let l be a line through A different from AB and suppose that X, Y, Z all lie on the same side of AB, as in the above diagram. Let perpendiculars from O_1, O_2, O_3 meet l at P, Q, R, respectively. Since a line through the centre of a circle bisects any chord,

$$AX = 2AP, \quad AY = 2AQ \quad \text{and} \quad AZ = 2AR.$$

Now

$$XY = AY - AX = 2(AQ - AP) = 2PQ \quad \text{and, similarly,} \quad YZ = 2QR.$$

Therefore $XY : YZ = PQ : QR$. But $O_1P \,||\, O_2Q \,||\, O_3R$, so $PQ : QR = O_1O_2 : O_2O_3$. Since the centres of the circles are fixed, the ratio $XY : YZ = O_1O_2 : O_2O_3$ does not depend on the choice of l.

If X, Y, Z do not all lie on the same side of AB, we can obtain the same result with a similar proof. For instance, if X and Y are opposite sides of AB, then we will have $XY = AY + AX$, but since in this case $PQ = AQ + AP$, it is still the case that $XY = 2PQ$ and result still follows, etc.

5. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Solution

We will construct the set T in the following way: Assume the points of S are in the xy-plane and let P be a point in S with maximum y-coordinate. This point P will be a member of the set T and now, from S, we will remove P and all points in S which are less than $\sqrt{3}$ units from P. From the remaining points we choose one with maximum y-coordinate to be a member of T and remove from S all points at distance less than $\sqrt{3}$ units from this new point. We continue in this way, until all the points of S are exhausted. Clearly any two points in T are at least $\sqrt{3}$ units apart. To show that T has at least $n/7$ points, we must prove that at each stage no more than 6 other points are removed along with P.

At a typical stage in this process, we've selected a point P with maximum y-coordinate, so any points at distance less than $\sqrt{3}$ from P must lie inside the semicircular region of radius $\sqrt{3}$ centred at P shown in the first diagram below. Since points of S are at least 1 unit apart, these points must lie outside (or on) the semicircle of radius 1. (So they lie in the shaded region of the first diagram.) Now divide this shaded region into 6 congruent regions $R_1, R_2, \ldots, R_6$ as shown in this diagram.

We will show that each of these regions contains at most one point of S. Since all 6 regions are congruent, consider one of them as depicted in the second diagram below. The distance between any two points in this shaded region must be less than the length of the line segment AB. The lengths of PA and PB are $\sqrt{3}$ and 1, respectively, and angle $APB = 30°$. If we construct a perpendicular from B to PA at C, then the length of PC is $\cos 30° = \sqrt{3}/2$. Thus BC is a perpendicular bisector of PA and therefore $AB = PB = 1$. So the distance between any two points in this region is less than 1. Therefore each of $R_1, \ldots, R_6$ can contain at most one point of S, which completes the proof.