2002 Canadian Mathematical Olympiad
Solutions

1. Let S be a subset of $\{1, 2, \ldots, 9\}$, such that the sums formed by adding each unordered pair of distinct numbers from S are all different. For example, the subset $\{1, 2, 3, 5\}$ has this property, but $\{1, 2, 3, 4, 5\}$ does not, since the pairs $\{1, 4\}$ and $\{2, 3\}$ have the same sum, namely 5.

   What is the maximum number of elements that S can contain?

   Solution 1

   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots, 9\}$ containing 6 elements such that all the sums of pairs are different. Now the smallest possible sum for two numbers from S is $1 + 2 = 3$ and the largest possible sum is $8 + 9 = 17$. That gives 15 possible sums: $3, \ldots, 17$. Also there are $\binom{6}{2} = 15$ pairs from S. Thus, each of $3, \ldots, 17$ is the sum of exactly one pair. The only pair from $\{1, \ldots, 9\}$ that adds to 3 is $\{1, 2\}$ and to 17 is $\{8, 9\}$. Thus $1, 2, 8, 9$ are in S. But then $1+9 = 2+8$, giving a contradiction. It follows that the maximum number of elements that S can contain is 5.

   Solution 2.

   It can be checked that all the sums of pairs for the set $\{1, 2, 3, 5, 8\}$ are different.

   Suppose, for a contradiction, that S is a subset of $\{1, \ldots 9\}$ such that all the sums of pairs are different and that $a_1 < a_2 < \ldots < a_6$ are the members of S.

   Since $a_1 + a_6 \neq a_2 + a_5$, it follows that $a_6 - a_5 \neq a_2 - a_1$. Similarly $a_6 - a_5 \neq a_4 - a_3$ and $a_4 - a_3 \neq a_2 - a_1$. These three differences must be distinct positive integers, so,

$$(a_6 - a_5) + (a_4 - a_3) + (a_2 - a_1) \geq 1 + 2 + 3 = 6\,.$$

   Similarly $a_3 - a_2 \neq a_5 - a_4$, so

$$(a_3 - a_2) + (a_5 - a_4) \geq 1 + 2 = 3\,.$$

   Adding the above 2 inequalities yields

$$a_6 - a_5 + a_5 - a_4 + a_4 - a_3 + a_3 - a_2 + a_2 - a_1 \geq 6 + 3 = 9\,,$$

   and hence $a_6 - a_1 \geq 9$. This is impossible since the numbers in S are between 1 and 9.

2. Call a positive integer n *practical* if every positive integer less than or equal to n can be written as the sum of distinct divisors of n.

   For example, the divisors of 6 are **1**, **2**, **3**, and **6** . Since

   1=**1**, 2=**2**, 3=**3**, 4=**1**+**3**, 5=**2**+ **3**, 6=**6**,

   we see that 6 is practical.

   Prove that the product of two practical numbers is also practical.

   Solution

   Let p and q be practical. For any $k \leq pq$, we can write

$$k = aq + b \text{ with } 0 \leq a \leq p,\ 0 \leq b < q.$$

   Since p and q are practical, we can write

$$a = c_1 + \ldots + c_m, \quad b = d_1 + \ldots + d_n$$

   where the c_i's are distinct divisors of p and the d_j's are distinct divisors of q. Now

$$\begin{aligned} k &= (c_1 + \ldots + c_m)q + (d_1 + \ldots + d_n) \\ &= c_1 q + \ldots + c_m q + d_1 + \ldots + d_n. \end{aligned}$$

   Each of $c_i q$ and d_j divides pq. Since $d_j < q \leq c_i q$ for any i, j, the $c_i q$'s and d_j's are all distinct, and we conclude that pq is practical.

3. Prove that for all positive real numbers a, b, and c,

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c,$$

and determine when equality occurs.

Each of the inequalities used in the solutions below has the property that equality holds if and only if $a = b = c$. Thus equality holds for the given inequality if and only if $a = b = c$.

Solution 1.

Note that $a^4 + b^4 + c^4 = \frac{(a^4 + b^4)}{2} + \frac{(b^4 + c^4)}{2} + \frac{(c^4 + a^4)}{2}$. Applying the arithmetic-geometric mean inequality to each term, we see that the right side is greater than or equal to

$$a^2b^2 + b^2c^2 + c^2a^2.$$

We can rewrite this as

$$\frac{a^2(b^2 + c^2)}{2} + \frac{b^2(c^2 + a^2)}{2} + \frac{c^2(a^2 + b^2)}{2}.$$

Applying the arithmetic mean-geometric mean inequality again we obtain $a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab$. Dividing both sides by abc (which is positive) the result follows.

Solution 2.

Notice the inequality is homogeneous. That is, if a, b, c are replaced by ka, kb, kc, $k > 0$ we get the original inequality. Thus we can assume, without loss of generality, that $abc = 1$. Then

$$\begin{aligned}\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} &= abc\left(\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab}\right) \\ &= a^4 + b^4 + c^4.\end{aligned}$$

So we need prove that $a^4 + b^4 + c^4 \geq a + b + c$.

By the Power Mean Inequality,

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a + b + c}{3}\right)^4,$$

so $a^4 + b^4 + c^4 \geq (a + b + c) \cdot \frac{(a + b + c)^3}{27}$.

By the arithmetic mean-geometric mean inequality, $\frac{a + b + c}{3} \geq \sqrt[3]{abc} = 1$, so $a + b + c \geq 3$.

Hence, $a^4 + b^4 + c^4 \geq (a + b + c) \cdot \frac{(a + b + c)^3}{27} \geq (a + b + c)\frac{3^3}{27} = a + b + c$.

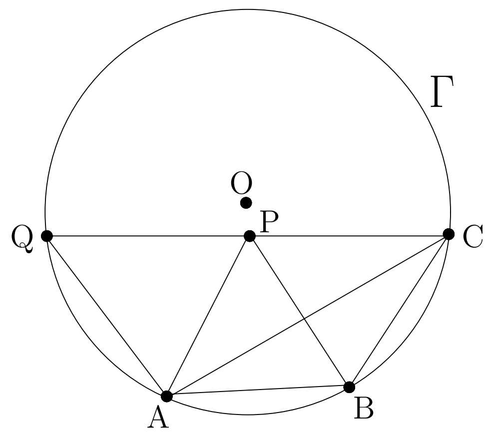
Solution 3.

Rather than using the Power-Mean inequality to prove $a^4 + b^4 + c^4 \geq a + b + c$ in Proof 2, the Cauchy-Schwartz-Bunjakovsky inequality can be used twice:

$$\begin{aligned}(a^4 + b^4 + c^4)(1^2 + 1^2 + 1^2) &\geq (a^2 + b^2 + c^2)^2 \\ (a^2 + b^2 + c^2)(1^2 + 1^2 + 1^2) &\geq (a + b + c)^2\end{aligned}$$

So $\frac{a^4 + b^4 + c^4}{3} \geq \frac{(a^2 + b^2 + c^2)^2}{9} \geq \frac{(a + b + c)^4}{81}$. Continue as in Proof 2.

4. Let Γ be a circle with radius r. Let A and B be distinct points on Γ such that $AB < \sqrt{3}r$. Let the circle with centre B and radius AB meet Γ again at C. Let P be the point inside Γ such that triangle ABP is equilateral. Finally, let CP meet Γ again at Q. Prove that $PQ = r$.



Solution 1.

Let the center of Γ be O, the radius r. Since $BP = BC$, let $\theta = \measuredangle BPC = \measuredangle BCP$.

Quadrilateral $QABC$ is cyclic, so $\measuredangle BAQ = 180° - \theta$ and hence $\measuredangle PAQ = 120° - \theta$.

Also $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 120° - \theta$, so $PQ = AQ$ and $\measuredangle AQP = 2\theta - 60°$.

Again because quadrilateral $QABC$ is cyclic, $\measuredangle ABC = 180° - \measuredangle AQC = 240° - 2\theta$.

Triangles OAB and OCB are congruent, since $OA = OB = OC = r$ and $AB = BC$.

Thus $\measuredangle ABO = \measuredangle CBO = \frac{1}{2}\measuredangle ABC = 120° - \theta$.

We have now shown that in triangles AQP and AOB, $\measuredangle PAQ = \measuredangle BAO = \measuredangle APQ = \measuredangle ABO$. Also $AP = AB$, so $\triangle AQP \cong \triangle AOB$. Hence $QP = OB = r$.

Solution 2.

Let the center of Γ be O, the radius r. Since A, P and C lie on a circle centered at B, $60° = \measuredangle ABP = 2\measuredangle ACP$, so $\measuredangle ACP = \measuredangle ACQ = 30°$.

Since Q, A, and C lie on Γ, $\measuredangle QOA = 2\measuredangle QCA = 60°$.

So $QA = r$ since if a chord of a circle subtends an angle of $60°$ at the center, its length is the radius of the circle.

Now $BP = BC$, so $\measuredangle BPC = \measuredangle BCP = \measuredangle ACB + 30°$.

Thus $\measuredangle APQ = 180° - \measuredangle APB - \measuredangle BPC = 90° - \measuredangle ACB$.

Since Q, A, B and C lie on Γ and $AB = BC$, $\measuredangle AQP = \measuredangle AQC = \measuredangle AQB + \measuredangle BQC = 2\measuredangle ACB$. Finally, $\measuredangle QAP = 180 - \measuredangle AQP - \measuredangle APQ = 90 - \measuredangle ACB$.

So $\measuredangle PAQ = \measuredangle APQ$ hence $PQ = AQ = r$.

5. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.

Solution 1.

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$; choose x, y such that $f(y) - f(x) > 0$ is minimal. Then

$$f(x) = \frac{xf(x) + yf(x)}{x + y} < \frac{xf(y) + yf(x)}{x + y} < \frac{xf(y) + yf(y)}{x + y} = f(y)$$

so $f(x) < f(x^2 + y^2) < f(y)$ and $0 < f(x^2 + y^2) - f(x) < f(y) - f(x)$, contradicting the choice of x and y. Thus, f is a constant function. Since $f(0)$ is in $\mathbb{N}$, the constant must be from $\mathbb{N}$.

Also, for any c in $\mathbb{N}$, $xc + yc = (x + y)c$ for all x and y, so $f(x) = c,\ c \in \mathbb{N}$ are the solutions to the equation.

Solution 2.

We claim f is a constant function. Define $g(x) = f(x) - f(0)$. Then $g(0) = 0,\ g(x) \geq -f(0)$ and

$$xg(y) + yg(x) = (x + y)g(x^2 + y^2)$$

for all x, y in $\mathbb{N}$.

Letting $y = 0$ shows $g(x^2) = 0$ (in particular, $g(1) = g(4) = 0$), and letting $x = y = 1$ shows $g(2) = 0$. Also, if x, y and z in $\mathbb{N}$ satisfy $x^2 + y^2 = z^2$, then

$$g(y) = -\frac{y}{x}g(x). \qquad (*)$$

Letting $x = 4$ and $y = 3$, $(*)$ shows that $g(3) = 0$.

For any even number $x = 2n > 4$, let $y = n^2 - 1$. Then $y > x$ and $x^2 + y^2 = (n^2 + 1)^2$. For any odd number $x = 2n+1 > 3$, let $y = 2(n+1)n$. Then $y > x$ and $x^2+y^2 = ((n+1)^2+n^2)^2$. Thus for every $x > 4$ there is $y > x$ such that $(*)$ is satisfied.

Suppose for a contradiction, that there is $x > 4$ with $g(x) > 0$. Then we can construct a sequence $x = x_0 < x_1 < x_2 < \ldots$ where $g(x_{i+1}) = -\dfrac{x_{i+1}}{x_i}\, g(x_i)$. It follows that $|g(x_{i+1})| > |g(x_i)|$ and the signs of $g(x_i)$ alternate. Since $g(x)$ is always an integer, $|g(x_{i+1})| \geq |g(x_i)|+1$. Thus for some sufficiently large value of i, $\ g(x_i) < -f(0)$, a contradiction.

As for Proof 1, we now conclude that the functions that satisfy the given functional equation are $f(x) = c,\ c \in \mathbb{N}$.

Solution 3. Suppose that W is the set of nonnegative integers and that $f : W \to W$ satisfies:

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2). \qquad (*)$$

We will show that f is a constant function.

Let $f(0) = k$, and set $S = \{x \mid f(x) = k\}$.

Letting $y = 0$ in $(*)$ shows that $f(x^2) = k \quad \forall\ x > 0$, and so

$$x^2 \in S \quad \forall\ x > 0 \qquad (1)$$

In particular, $1 \in S$.

Suppose $x^2 + y^2 = z^2$. Then $yf(x) + xf(y) = (x + y)f(z^2) = (x + y)k$. Thus,

$$x \in S \quad \text{iff} \quad y \in S. \tag{2}$$

whenever $x^2 + y^2$ is a perfect square.

For a contradiction, let n be the smallest non-negative integer such that $f(2^n) \neq k$. By (1) n must be odd, so $\dfrac{n-1}{2}$ is an integer. Now $\dfrac{n-1}{2} < n$ so $f(2^{\frac{n-1}{2}}) = k$. Letting $x = y = 2^{\frac{n-1}{2}}$ in $(*)$ shows $f(2^n) = k$, a contradiction. Thus every power of 2 is an element of S.

For each integer $n \geq 2$ define $p(n)$ to be the *largest prime* such that $p(n) \mid n$.

Claim: For any integer $n > 1$ that is not a power of 2, there exists a sequence of integers $x_1, x_2, \ldots, x_r$ such that the following conditions hold:

a) $x_1 = n$.
b) $x_i^2 + x_{i+1}^2$ is a perfect square for each $i = 1, 2, 3, \ldots, r - 1$.
c) $p(x_1) \geq p(x_2) \geq \ldots \geq p(x_r) = 2$.

Proof: Since n is not a power of 2, $p(n) = p(x_1) \geq 3$. Let $p(x_1) = 2m + 1$, so $n = x_1 = b(2m + 1)^a$, for some a and b, where $p(b) < 2m + 1$.

Case 1: $a = 1$. Since $(2m+1, 2m^2+2m, 2m^2+2m+1)$ is a Pythagorean Triple, if $x_2 = b(2m^2 + 2m)$, then $x_1^2 + x_2^2 = b^2(2m^2 + 2m + 1)^2$ is a perfect square. Furthermore, $x_2 = 2bm(m + 1)$, and so $p(x_2) < 2m + 1 = p(x_1)$.

Case 2: $a > 1$. If $n = x_1 = (2m + 1)^a \cdot b$, let $x_2 = (2m + 1)^{a-1} \cdot b \cdot (2m^2 + 2m)$, $x_3 = (2m+1)^{a-2} \cdot b \cdot (2m^2+2m)^2$, …, $x_{a+1} = (2m+1)^0 \cdot b \cdot (2m^2+2m)^a = b \cdot 2^a m^a (m+1)^a$. Note that for $1 \leq i \leq a$, $x_i^2 + x_{i+1}^2$ is a perfect square and also note that $p(x_{a+1}) < 2m+1 = p(x_1)$.

If x_{a+1} is not a power of 2, we extend the sequence x_i using the same procedure described above. We keep doing this until $p(x_r) = 2$, for some integer r.

By (2), $x_i \in S$ iff $x_{i+1} \in S$ for $i = 1, 2, 3, \ldots, r - 1$. Thus, $n = x_1 \in S$ iff $x_r \in S$. But x_r is a power of 2 because $p(x_r) = 2$, and we earlier proved that powers of 2 are in S. Therefore, $n \in S$, proving the claim.

We have proven that every integer $n \geq 1$ is an element of S, and so we have proven that $f(n) = k = f(0)$, for each $n \geq 1$. Therefore, f is constant, Q.E.D.