CMO 1996
SOLUTIONS

QUESTION 1

Solution.

If \( f(x) = x^3 - x - 1 = (x - \alpha)(x - \beta)(x - \gamma) \) has roots \( \alpha, \beta, \gamma \) standard results about roots of polynomials give \( \alpha + \beta + \gamma = 0 \), \( \alpha \beta + \alpha \gamma + \beta \gamma = -1 \), and \( \alpha \beta \gamma = 1 \).

Then

\[
S = \frac{1 + \alpha}{1 - \alpha} + \frac{1 + \beta}{1 - \beta} + \frac{1 + \gamma}{1 - \gamma} = \frac{N}{(1 - \alpha)(1 - \beta)(1 - \gamma)}
\]

where the numerator simplifies to

\[
N = 3 - (\alpha + \beta + \gamma) - (\alpha \beta + \alpha \gamma + \beta \gamma) + 3 \alpha \beta \gamma
\]

\[
= 3 - (0) - (-1) + 3(1)
\]

\[= 7.\]

The denominator is \( f(1) = -1 \) so the required sum is \(-7\).
QUESTION 2

Solution 1.

For any $t$, $0 \leq 4t^2 < 1 + 4t^2$, so $0 \leq \frac{4t^2}{1 + 4t^2} < 1$. Thus $x$, $y$ and $z$ must be non-negative and less than 1.

Observe that if one of $x$ $y$ or $z$ is 0, then $x = y = z = 0$.

If two of the variables are equal, say $x = y$, then the first equation becomes

$$\frac{4x^2}{1 + 4x^2} = x.$$ 

This has the solution $x = 0$, which gives $x = y = z = 0$ and $x = \frac{1}{2}$ which gives $x = y = z = \frac{1}{2}$.

Finally, assume that $x$, $y$ and $z$ are non-zero and distinct. Without loss of generality we may assume that either $0 < x < y < z < 1$ or $0 < x < z < y < 1$. The two proofs are similar, so we do only the first case.

We will need the fact that $f(t) = \frac{4t^2}{1 + 4t^2}$ is increasing on the interval $(0, 1)$.

To prove this, if $0 < s < t < 1$ then

$$f(t) - f(s) = \frac{4t^2}{1 + 4t^2} - \frac{4s^2}{1 + 4s^2} = \frac{4t^2 - 4s^2}{(1 + 4s^2)(1 + 4t^2)} > 0.$$ 

So $0 < x < y < z \Rightarrow f(x) = y < f(y) = z < f(z) = x$, a contradiction.

Hence $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only real solutions.

Solution 2.

Notice that $x$, $y$ and $z$ are non-negative. Adding the three equations gives

$$x + y + z = \frac{4z^2}{1 + 4z^2} + \frac{4x^2}{1 + 4x^2} + \frac{4y^2}{1 + 4y^2}.$$ 

This can be rearranged to give

$$\frac{x(2x - 1)^2}{1 + 4x^2} + \frac{y(2y - 1)^2}{1 + 4y^2} + \frac{z(2z - 1)^2}{1 + 4z^2} = 0.$$ 

Since each term is non-negative, each term must be 0, and hence each variable is either 0 or $\frac{1}{2}$. The original equations then show that $x = y = z = 0$ and $x = y = z = \frac{1}{2}$ are the only two solutions.
Solution 3.

Notice that $x, y, \text{ and } z$ are non-negative. Multiply both sides of the inequality

$$\frac{y}{1 + 4y^2} \geq 0$$

by $(2y - 1)^2$, and rearrange to obtain

$$y - \frac{4y^2}{1 + 4y^2} \geq 0,$$

and hence that $y \geq z$. Similarly, $z \geq x$, and $x \geq y$. Hence, $x = y = z$ and, as in Solution 1, the two solutions follow.

Solution 4.

As for solution 1, note that $x = y = z = 0$ is a solution and any other solution will have each of $x, y \text{ and } z$ positive.

The arithmetic-geometric mean inequality (or direct computation) shows that $\frac{1 + 4x^2}{2} \geq \sqrt{1 \cdot 4x^2} = 2x$
and hence $x \geq \frac{4x^2}{1 + 4x^2} = y$, with equality if and only if $1 = 4x^2$ – that is, $x = \frac{1}{2}$. Similarly, $y \geq z$
with equality if and only if $y = \frac{1}{2}$ and $z \geq x$ with equality if and only if $z = \frac{1}{2}$. Adding $x \geq y, \ y \geq z$
and $z \geq x$ gives $x + y + x \geq x + y + z$. Thus equality must occur in each inequality, so $x = y = z = \frac{1}{2}$. 
QUESTION 3

Solution.

Let $a_1, a_2, \ldots, a_n$ be a permutation of $1, 2, \ldots, n$ with properties (i) and (ii).

A crucial observation, needed in Case II (b) is the following: If $a_k$ and $a_{k+1}$ are consecutive integers (i.e. $a_{k+1} = a_k \pm 1$), then the terms to the right of $a_{k+1}$ (also to the left of $a_k$) are either all less than both $a_k$ and $a_{k+1}$ or all greater than both $a_k$ and $a_{k+1}$.

Since $a_1 = 1$, by (ii) $a_2$ is either 2 or 3.

**CASE I:** Suppose $a_2 = 2$. Then $a_3, a_4, \ldots, a_n$ is a permutation of $3, 4, \ldots, n$. Thus $a_2, a_3, \ldots, a_n$ is a permutation of $2, 3, \ldots, n$ with $a_2 = 2$ and property (ii). Clearly there are $f(n - 1)$ such permutations.

**CASE II:** Suppose $a_2 = 3$.

(a) Suppose $a_3 = 2$. Then $a_4, a_5, \ldots, a_n$ is a permutation of $4, 5, \ldots, n$ with $a_4 = 4$ and property (ii). There are $f(n - 3)$ such permutations.

(b) Suppose $a_3 \geq 4$. If $a_{k+1}$ is the first even number in the permutation then, because of (ii), $a_1, a_2, \ldots, a_k$ must be $1, 3, 5, \ldots, 2k - 1$ (in that order). Then $a_{k+1}$ is either $2k$ or $2k - 2$, so that $a_k$ and $a_{k+1}$ are consecutive integers. Applying the crucial observation made above, we deduce that $a_{k+2}, \ldots, a_n$ are all either greater than or smaller than $a_k$ and $a_{k+1}$. But 2 must be to the right of $a_{k+1}$. Hence $a_{k+2}, \ldots, a_n$ are the even integers less than $a_{k+1}$. The only possibility then, is

$$1, 3, 5, \ldots, a_{k-1}, a_k, \ldots, 6, 4, 2.$$ 

Cases I and II show that

$$f(n) = f(n - 1) + f(n - 3) + 1, \quad n \geq 4. \quad (*)$$

Calculating the first few values of $f(n)$ directly gives

$$f(1) = 1, \quad f(2) = 1, \quad f(3) = 2, \quad f(4) = 4, \quad f(5) = 6.$$ 

Calculating a few more $f(n)$’s using (*) and mod 3 arithmetic, $f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 1, f(5) = 0, f(6) = 0, f(7) = 2, f(8) = 0, f(9) = 1, f(10) = 1, f(11) = 2$. Since $f(1) = f(9), f(2) = f(10)$ and $f(3) = f(11)$ mod 3, (*) shows that $f(a) = f(a \mod 8)$, mod 3, $a \geq 1$.

Hence $f(1996) \equiv f(4) \equiv 1 \pmod{3}$ so 3 does not divide $f(1996)$. 


QUESTION 4

Solution 1.

Let $BE = BD$ with $E$ on $BC$, so that $AD = EC$:

By a standard theorem, $\frac{AB}{CB} = \frac{AD}{DC}$; so in

$\triangle CED$ and $\triangle CAB$ we have a common angle and

$$\frac{CE}{CD} = \frac{AD}{CD} = \frac{AB}{CB} = \frac{CA}{CB}.$$ 

Thus $\triangle CED \sim \triangle CAB$, so that $\angle CDE = \angle DCE = \angle ABC = 2x$.

Hence $\angle BDE = \angle BED = 4x$, whence $9x = 180^\circ$ so $x = 20^\circ$.

Thus $\angle A = 180^\circ - 4x = 100^\circ$.

Solution 2.

Apply the law of sines to $\triangle ABD$ and $\triangle BDC$ to get

$$\frac{AD}{BD} = \frac{\sin x}{\sin 4x} \quad \text{and} \quad 1 + \frac{AD}{BD} = \frac{BC}{BD} = \frac{\sin 3x}{\sin 2x}.$$ 

Now massage the resulting trigonometric equation with standard identities to get

$$\sin 2x (\sin 4x + \sin x) = \sin 2x (\sin 5x + \sin x).$$ 

Since $0 < 2x < 90^\circ$, we get

$$5x - 90^\circ = 90^\circ - 4x,$$

so that $\angle A = 100^\circ$. 
QUESTION 5

Solution.
Let
\[
f(n) = n - \sum_{k=1}^{m} [r_k n]
\]
\[= n \sum_{k=1}^{m} r_k - \sum_{k=1}^{m} [r_k n]
\]
\[= \sum_{k=1}^{m} \{r_k n - [r_k n]\}.
\]

Now \(0 \leq x - [x] < 1\), and if \(c\) is an integer, \((c + x) - [c + x] = x - [x]\).

Hence \(0 \leq f(n) < \sum_{k=1}^{m} 1 = m\). Because \(f(n)\) is an integer, \(0 \leq f(n) \leq m - 1\).

To show that \(f(n)\) can achieve these bounds for \(n > 0\), we assume that \(r_k = \frac{a_k}{b_k}\) where \(a_k, b_k\) are integers; \(a_k < b_k\).

Then, if \(n = b_1 b_2 \ldots b_m\), \((r_k n) - [r_k n] = 0\), \(k = 1, 2, \ldots, m\) and thus \(f(n) = 0\).

Letting \(n = b_1 b_2 \ldots b_n - 1\), then
\[
r_k n = r_k (b_1 b_2 \ldots b_n - 1)
\]
\[= r_k \{(b_1 b_2 \ldots b_m - b_k) + b_k - 1\}
\]
\[= \text{integer} + r_k (b_k - 1).
\]

This gives
\[
r_k n - [r_k n] = r_k (b_k - 1) - [r_k (b_k - 1)]
\]
\[= \frac{a_k}{b_k} (b_k - 1) - \left[\frac{a_k}{b_k} (b_k - 1)\right]
\]
\[= \left(a_k - \frac{a_k}{b_k}\right) - \left[a_k - \frac{a_k}{b_k}\right]
\]
\[= \left(a_k - \frac{a_k}{b_k}\right) - (a_k - 1)
\]
\[= 1 - \frac{a_k}{b_k} = 1 - r_k.
\]

Hence
\[f(n) = \sum_{k=1}^{m} (1 - r_k) = m - 1.
\]