SOLUTIONS

QUESTION 1

Solution 1.

Let $S$ denote the given sum. Then

$$S = \sum_{n=1}^{1994} (-1)^n \left( \frac{n}{(n-1)!} + \frac{n+1}{n!} \right)$$

$$= \sum_{n=0}^{1993} (-1)^{n+1} \frac{n+1}{n!} + \sum_{n=1}^{1994} (-1)^n \frac{n+1}{n!}$$

$$= -1 + \frac{1995}{1994!}$$

Solution 2.

For positive integers $k$, define

$$S(k) = \sum_{n=1}^{k} (-1)^n \frac{n^2 + n + 1}{n!}.$$

We prove by induction on $k$ that

(*) \quad S(k) = -1 + (-1)^{k+1} \frac{k+1}{k!}.

The given sum is the case when $k = 1994$. For $k = 1$, $S(1) = -3 = -1 - \frac{2}{1!}$. Suppose (*) holds for some $k \geq 1$, then

$$S(k+1) = S(k) + (-1)^{k+1} \frac{(k+1)^2 + (k+1) + 1}{(k+1)!}$$

$$= -1 + (-1)^k \frac{k+1}{k!} + (-1)^{k+1} \left( \frac{k+1}{k!} + \frac{k+2}{(k+1)!} \right)$$

$$= -1 + (-1)^{k+1} \frac{k+2}{(k+1)!}$$

completing the induction.
SOLUTIONS (Cont'd)

QUESTION 2

Solution 1.

Fix a positive integer \( n \). Let \( a = (\sqrt{2} - 1)^n \) and \( b = (\sqrt{2} + 1)^n \). Then clearly \( ab = 1 \). Let \( c = (b + a)/2 \) and \( d = (b - a)/2 \). If \( n \) is even, \( n = 2k \), then from the Binomial Theorem we get

\[
c = \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} (\sqrt{2}^{n-i} + (-1)^i \sqrt{2}^{n-i})
= \sum_{j=0}^{k} \binom{2k}{2j} \sqrt{2}^{2k-2j}
= \sum_{j=0}^{k} \binom{2k}{2j} 2^{k-j}
\]

(1)

and

\[
d = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \sum_{i=0}^{n} \binom{n}{i} (\sqrt{2}^{n-i} - (-1)^i \sqrt{2}^{n-i})
= \frac{2}{\sqrt{2}} \sum_{j=0}^{k-1} \binom{2k}{2j + 1} \sqrt{2}^{2k-2j-1}
= \sum_{j=0}^{k-1} \binom{2k}{2j + 1} 2^{k-j}
\]

(2)

showing that \( c \) and \( d \) are both positive integers. Similarly, when \( n \) is odd we see that \( c \) and \( d \) are both positive integers. In either case, \( c^2 \) and \( d^2 \) are both integers. Notes that

\[
c^2 - d^2 = \frac{1}{4}((b + a)^2 - (b - a)^2) = ab = 1.
\]

Hence if we let \( m = c^2 \), then \( m - 1 = c^2 - 1 = d^2 \) and \( a = c - d = \sqrt{m} - \sqrt{m - 1} \).

1994 Canadian Mathematical Olympiad
- 13 -
Solution 2.

Let \( m \) and \( n \) be positive integers. Observe that

\[
(\sqrt{2} - 1)^n(\sqrt{2} + 1)^n = 1 = (\sqrt{m} - \sqrt{m - 1})(\sqrt{m} + \sqrt{m - 1})
\]

and so

\[
(*) \quad (\sqrt{2} - 1)^n = \sqrt{m} - \sqrt{m - 1} \quad \text{if and only if} \quad (\sqrt{2} + 1)^n = \sqrt{m} + \sqrt{m - 1}.
\]

Assuming \( m \) and \( n \) satisfy (*), then adding the two equivalent equations we get

\[
2\sqrt{m} = (\sqrt{2} - 1)^n + (\sqrt{2} + 1)^n \quad \text{whence:}
\]

\[
(**) \quad m = \frac{1}{4}[(\sqrt{2} - 1)^{2n} + 2 + (\sqrt{2} + 1)^{2n}].
\]

Now we show that the steps above are reversible and that \( m \) defined by (**) is a positive integer. From (**), one sees easily that

\[
\sqrt{m} = \frac{1}{2}[(\sqrt{2} - 1)^n + (\sqrt{2} + 1)^n] \quad \text{and} \quad \sqrt{m - 1} = \frac{1}{2}[(\sqrt{2} + 1)^n - (\sqrt{2} - 1)^n],
\]

and so \( \sqrt{m} - \sqrt{m - 1} = (\sqrt{2} - 1)^n \) as required. Finally, from the Binomial Theorem,

\[
(\sqrt{2} - 1)^{2n} + (\sqrt{2} + 1)^{2n} =
\]

\[
= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k 2^{(2n-k)/2} + 2^{(2n-k)/2}
\]

\[
= \sum_{\ell=0}^{n} \binom{2n}{2\ell} 2^{n-\ell+1}
\]

which is congruent to 2 modulo 4 since \( 2^{n-\ell+1} \equiv 0 \pmod{4} \) for all \( \ell = 0, 1, 2, \ldots, n - 1 \). Therefore, \( (\sqrt{2} - 1)^{2n} + 2 + (\sqrt{2} + 1)^{2n} \) is a multiple of 4, as required.
Solution 3.

We show by induction that

\[(\sqrt{2} - 1)^n = \begin{cases} a\sqrt{2} - b & \text{where } 2a^2 = b^2 + 1 \text{ if } n \text{ is odd} \\ a - b\sqrt{2} & \text{where } a^2 = 2b^2 + 1 \text{ if } n \text{ is even} \end{cases} \]

Thus \( m = 2a^2 \) when \( n \) is odd and \( m = a^2 \) when \( n \) is even and the problem is solved.

The induction is as follows:

\[(\sqrt{2} - 1)^1 = 1\sqrt{2} - 1 \text{ where } 2(1^2) = 1^2 + 1 \]
\[(\sqrt{2} - 1)^2 = 3 - 2\sqrt{2} \text{ where } 3^2 = 2(2^2) + 1. \]

Assume \((*)\) holds for some \( n \geq 1, n \) odd. Then

\[(\sqrt{2} - 1)^{n+1} = (a\sqrt{2} - b)(\sqrt{2} - 1) \text{ where } 2a^2 = b^2 + 1 \]
\[= (2a + b) - (a + b)\sqrt{2} \]
\[= A - B\sqrt{2} \text{ where } A = 2a + b, B = a + b. \]

Moreover, \( A^2 = 2a^2 + 4ab + b^2 + 2a^2 = 2a^2 + 4ab + 2b^2 + 1 = 2B^2 + 1. \)

Assume \((*)\) holds for some \( n \geq 2, n \) even. Then

\[(\sqrt{2} - 1)^{n+1} = (a - b\sqrt{2})(\sqrt{2} - 1) \text{ where } a^2 = 2b^2 + 1 \]
\[= (a + b)\sqrt{2} - (a + 2b) \]
\[= A\sqrt{2} - B \text{ where } A = a + b, B = a + 2b. \]

Moreover, \( 2A^2 = 2a^2 + 4ab + 2b^2 = a^2 + 4ab + 4b^2 + a^2 - 2b^2 = B^2 + 1. \)
Solution 4.

From \((\sqrt{2} - 1)^1 = \sqrt{2} - 1, (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}, (\sqrt{2} - 1)^3 = 5\sqrt{2} - 7, (\sqrt{2} - 1)^4 = 17 - 12\sqrt{2}\), etc, we conjecture that

\[
(*) \quad (\sqrt{2} - 1)^n = s_n\sqrt{2} + t_n
\]

where \(s_1 = 1, t_1 = 1, s_{n+1} = (-1)^n(|s_n| + |t_n|), t_{n+1} = (-1)^{n+1}(2|s_n| + |t_n|)\).

Note that \(s_n\) is positive (negative) if \(n\) is odd (even) and \(t_n\) is negative (positive) if \(n\) is odd (even).

We now show by induction that (*) holds and that each \(s_n\sqrt{2} + t_n\) of the form \(\sqrt{m} - \sqrt{m - 1}\) for some \(m\).

It is easily verified that (*) is correct for \(n = 1\) and \(2\). Assume (*) holds for some \(n \geq 2\). Then

\[
(\sqrt{2} - 1)^{n+1} = (s_n\sqrt{2} + t_n)(\sqrt{2} - 1) = (t_n - s_n)\sqrt{2} + (2s_n - t_n).
\]

If \(n\) is odd, then

\[
t_n - s_n = -(|t_n| + |s_n|) = s_{n+1} \\
2s_n - t_n = 2|s_n| + |t_n| = t_{n+1}.
\]

If \(n\) is even, then

\[
t_n - s_n = |t_n| + |s_n| = s_{n+1} \\
2s_n - t_n = -2|s_n| - |t_n| = t_{n+1}.
\]

We have shown that (*) is correct for all \(n\).

Observe now that \((s_{n+1}\sqrt{2})^2 - t_{n+1}^2 = 2(s_n^2 - 2s_nt_n + t_n^2) - (4s_n^2 - 4s_nt_n + t_n^2) = -2s_n^2 + t_n^2 = -(s_n\sqrt{2})^2 - t_n^2\). Since \((s_1\sqrt{2})^2 - t_1^2 = 1\), it follows that \((s_n\sqrt{2})^2 - t_n^2 = (-1)^{n+1}\) for all \(n\). To complete the proof it suffices to take \(m = (s_n\sqrt{2})^2, m - 1 = t_n^2\) when \(n\) is odd and \(m = t_n^2, m - 1 = (s_n\sqrt{2})^2\) when \(n\) is even.
SOLUTIONS (Cont'd)

QUESTION 3

First observe that if two neighbours have the same response on the \( n^{th} \) vote, then they both will respond the same way on the \((n + 1)^{th}\) vote. Moreover, neither will ever change his response after the \( n^{th} \) vote.

Let \( A_n \) be the set of men who agree with at least one of their neighbours on the \( n^{th} \) vote. The previous paragraph says that \( A_n \subseteq A_{n+1} \) for every \( n \geq 1 \). Moreover, we will be done if we can show that \( A_n \) contains all 25 men for some \( n \).

Since there are an odd number of men at the table, it is not possible that every man disagrees with both of his neighbours on the first vote. Therefore \( A_1 \) contains at least two men. And since \( A_n \subseteq A_{n+1} \) for every \( n \), there exists a \( T < 25 \) such that \( A_T = A_{T+1} \). Suppose that \( A_T \) does not contain all 25 men; we shall use this to derive a contradiction. Since \( A_T \) is not empty, there must exist two neighbours, whom we shall call \( x \) and \( y \), such that \( x \in A_T \) and \( y \not\in A_T \). Since \( x \in A_T \), he will respond the same way on the \( T^{th} \) and \((T + 1)^{th}\) votes. But \( y \not\in A_T \), so \( y \)'s response on the \( T^{th} \) vote differs from \( x \)'s response. In fact, we know that \( y \) disagrees with both of his neighbours on the \( T^{th} \) vote, and so he will change his response on the \((T + 1)^{th}\) vote. Therefore, on the \((T + 1)^{th}\) vote, \( y \) responds the same way as does \( x \). This implies that \( y \in A_{T+1} \). But \( y \not\in A_T \), which contradicts the fact that \( A_T = A_{T+1} \). Therefore we conclude that \( A_T \) contains all 25 men, and we are done.

QUESTION 4

There are three cases to be considered:

Case 1: If \( P \) is outside \( \Omega \) (see figures I, II, and III), then since \( \angle AUB = \angle AVB = \pi/2 \), we have

\[
\cos(\angle APB) = \frac{PU}{PB} = \frac{PV}{PA} = \sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = \sqrt{3}\text{.}
\]

![Figure I](image1)

![Figure II](image2)

![Figure III](image3)

1994 Canadian Mathematical Olympiad
- 17 -
Case 2: If \( P \) is on \( \Omega \) (see figure IV), then

\[
P = U = V \Rightarrow PU = PV = 0 \Rightarrow s = t = 0.
\]

Since \( \angle APB = \pi/2 \), \( \cos(\angle APB) = 0 = st \) holds again.

\[\text{Figure IV} \quad \quad \quad \text{Figure V}\]

Case 3: If \( P \) is inside \( \Omega \) (figure V), then

\[
\cos(\angle APB) = \cos(\pi - \angle APV) = -\cos(\angle APV) = -\frac{PV}{PA},
\]

and

\[
\cos(\angle APB) = \cos(\pi - \angle BPU) = -\cos(\angle BPU) = -\frac{PU}{PB}.
\]

Therefore \( \cos(\angle APB) = -\sqrt{\frac{PV}{PA} \cdot \frac{PV}{PB}} = -\sqrt{st}. \)
QUESTION 5

Solution 1.

From A draw a line l parallel to BC. Extend DF and DE to meet l at P and Q respectively (See Figure 1). Then from similar triangles, we have

$$\frac{AP}{BD} = \frac{AF}{FB}$$

and

$$\frac{AQ}{CD} = \frac{AE}{EC}$$

or

$$AP = \frac{AF}{FB} \cdot BD$$ and $$AQ = \frac{AE}{EC} \cdot CD.$$  \hfill (1)

By Ceva's Theorem, $$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$ and thus

$$\frac{AF}{FB} \cdot BD = \frac{AE}{EC} \cdot CD$$  \hfill (2)

From (1) and (2) we get $$AP = AQ$$ and hence $$\triangle ADP \cong \triangle ADQ$$ from which $$\angle EDH = \angle FDH$$ follows.

![Figure 1](image)

Solution 2.

Use cartesian coordinates, with D at (0,0), A = (0,a), B = (-b,0), C = (c,0). Let H = (0,h), E = (u,v) and F = (-r,s) where a,b,c,h,u,v,r,s are all positive (See Figure II).
SOLUTIONS (Cont'd)

It clearly suffices to show that \( \frac{v}{u} = \frac{\delta}{r} \). Since \( EC \) and \( AC \) have the same slope, we have
\[
\frac{v}{u-c} = \frac{a}{c}.
\]
Similarly, since \( EB \) and \( HB \) have the same slope, \( \frac{v}{u+b} = \frac{k}{b} \). Thus
\[
\frac{v}{a} = \frac{u-c}{-c} = \frac{-u}{c} + 1 \quad (1)
\]
and
\[
\frac{v}{h} = \frac{u+b}{b} = \frac{u}{b} + 1 \quad (2)
\]

(2)-(1) we get
\[
\frac{v}{h} = \frac{u}{b} + 1 = \frac{ah(b+c)}{bc(a-h)}.
\]

With \( u, v, b \) and \( c \) replaced by \(-r, s, -c\) and \(-b\) respectively, we have, by a similar argument that
\[
\frac{s}{-r} = \frac{ah(-c-b)}{bc(a-h)} \quad \text{or} \quad \frac{s}{r} = \frac{ah(b+c)}{bc(a-h)}.
\]

Therefore, \( \frac{v}{u} = \frac{\delta}{r} \) as desired.