SOLUTIONS

QUESTION 1

Solution 1.

Let $\mathcal{S}$ denote the given sum. Then

$$\begin{aligned}\mathcal{S} &= \sum_{n=1}^{1994}(-1)^n\left(\frac{n}{(n-1)!}+\frac{n+1}{n!}\right)\\ &= \sum_{n=0}^{1993}(-1)^{n+1}\frac{n+1}{n!}+\sum_{n=1}^{1994}(-1)^n\frac{n+1}{n!}\\ &= -1+\frac{1995}{1994!}\end{aligned}$$

Solution 2.

For positive integers k, define

$$\mathcal{S}(k)=\sum_{n=1}^{k}(-1)^n\frac{n^2+n+1}{n!}.$$

We prove by induction on k that

$$(*) \qquad \mathcal{S}(k) = -1+(-1)^k\frac{k+1}{k!}.$$

The given sum is the case when $k = 1994$. For $k = 1$, $\mathcal{S}(1) = -3 = -1 - \frac{2}{1!}$. Suppose (*) holds for some $k \geq 1$, then

$$\begin{aligned}\mathcal{S}(k+1) &= \mathcal{S}(k)+(-1)^{k+1}\frac{(k+1)^2+(k+1)+1}{(k+1)!}\\ &= -1+(-1)^k\frac{k+1}{k!}+(-1)^{k+1}\left(\frac{k+1}{k!}+\frac{k+2}{(k+1)!}\right)\\ &= -1+(-1)^{k+1}\frac{k+2}{(k+1)!}\end{aligned}$$

completing the induction.

QUESTION 2

Solution 1.

Fix a positive integer n. Let $a = (\sqrt{2}-1)^n$ and $b = (\sqrt{2}+1)^n$. Then clearly $ab = 1$. Let $c = (b+a)/2$ and $d = (b-a)/2$. If n is even, $n = 2k$, then from the Binomial Theorem we get

$$\begin{aligned} c &= \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\left(\sqrt{2}^{n-i} + (-1)^i\sqrt{2}^{n-i}\right) \\ &= \sum_{j=0}^{k}\binom{2k}{2j}\sqrt{2}^{2k-2j} \\ &= \sum_{j=0}^{k}\binom{2k}{2j}2^{k-j} \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{d}{\sqrt{2}} &= \frac{1}{\sqrt{2}}\sum_{i=0}^{n}\binom{n}{i}\left(\sqrt{2}^{n-i} - (-1)^i\sqrt{2}^{n-i}\right) \\ &= \frac{2}{\sqrt{2}}\sum_{j=0}^{k-1}\binom{2k}{2j+1}\sqrt{2}^{2k-2j-1} \\ &= \sum_{j=0}^{k-1}\binom{2k}{2j+1}2^{k-j} \end{aligned} \tag{2}$$

showing that c and $\frac{d}{\sqrt{2}}$ are both positive integers. Similarly, when n is odd we see that $\frac{c}{\sqrt{2}}$ and d are both positive integers. In either case, c^2 and d^2 are both integers. Notes that

$$c^2 - d^2 = \frac{1}{4}((b+a)^2 - (b-a)^2) = ab = 1.$$

Hence if we let $m = c^2$, then $m - 1 = c^2 - 1 = d^2$ and $a = c - d = \sqrt{m} - \sqrt{m-1}$.

Solution 2.

Let m and n be positive integers. Observe that

$$(\sqrt{2}-1)^n(\sqrt{2}+1)^n = 1 = (\sqrt{m}-\sqrt{m-1})(\sqrt{m}+\sqrt{m-1})$$

and so

$$(*) \quad (\sqrt{2}-1)^n = \sqrt{m}-\sqrt{m-1} \text{ if and only if } (\sqrt{2}+1)^n = \sqrt{m}+\sqrt{m-1}.$$

Assuming m and n satisfy (*), then adding the two equivalent equations we get $2\sqrt{m} = (\sqrt{2}-1)^n + (\sqrt{2}+1)^n$ whence:

$$(**) \qquad m = \frac{1}{4}[(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}].$$

Now we show that the steps above are reversible and that m defined by (**) is a positive integer. From (**) one sees easily that

$$\sqrt{m} = \frac{1}{2}[(\sqrt{2}-1)^n + (\sqrt{2}+1)^n] \text{ and } \sqrt{m-1} = \frac{1}{2}[(\sqrt{2}+1)^n - (\sqrt{2}-1)^n],$$

and so $\sqrt{m}-\sqrt{m-1} = (\sqrt{2}-1)^n$ as required. Finally, from the Binomial Theorem,

$$\begin{aligned}(\sqrt{2}-1)^{2n} + (\sqrt{2}+1)^{2n} &= \\ &= \sum_{k=0}^{2n} \binom{2n}{k} [(-1)^k 2^{(2n-k)/2} + 2^{(2n-k)/2}] \\ &= \sum_{\ell=0}^{n} \binom{2n}{2\ell} 2^{n-\ell+1}\end{aligned}$$

which is congruent to 2 modulo 4 since $2^{n-\ell+1} \equiv 0 \pmod 4$ for all $\ell = 0, 1, 2, \cdots, n-1$. Therefore, $(\sqrt{2}-1)^{2n} + 2 + (\sqrt{2}+1)^{2n}$ is a multiple of 4, as required.

Solution 3.

We show by induction that

$$(*)\qquad (\sqrt{2}-1)^n = \begin{cases} a\sqrt{2}-b \text{ where } 2a^2 = b^2+1 \text{ if } n \text{ is odd} \\ a-b\sqrt{2} \text{ where } a^2 = 2b^2+1 \text{ if } n \text{ is even} \end{cases}.$$

Thus $m = 2a^2$ when n is odd and $m = a^2$ when n is even and the problem is solved.

The induction is as follows:

$$(\sqrt{2}-1)^1 = 1\sqrt{2}-1 \text{ where } 2(1^2) = 1^2+1$$
$$(\sqrt{2}-1)^2 = 3-2\sqrt{2} \text{ where } 3^2 = 2(2^2)+1.$$

Assume (*) holds for some $n \geq 1$, n odd. Then

$$\begin{aligned}&(\sqrt{2}-1)^{n+1}\\ &\quad = (a\sqrt{2}-b)(\sqrt{2}-1) \text{ where } 2a^2 = b^2+1\\ &\quad = (2a+b)-(a+b)\sqrt{2}\\ &\quad = A-B\sqrt{2} \text{ where } A = 2a+b, B = a+b.\end{aligned}$$

Moreover, $A^2 = 2a^2+4ab+b^2+2a^2 = 2a^2+4ab+2b^2+1 = 2B^2+1$.

Assume (*) holds for some $n \geq 2$, n even. Then

$$\begin{aligned}&(\sqrt{2}-1)^{n+1}\\ &\quad = (a-b\sqrt{2})(\sqrt{2}-1) \text{ where } a^2 = 2b^2+1\\ &\quad = (a+b)\sqrt{2}-(a+2b)\\ &\quad = A\sqrt{2}-B \text{ where } A = a+b, B = a+2b.\end{aligned}$$

Moreover, $2A^2 = 2a^2+4ab+2b^2 = a^2+4ab+4b^2+a^2-2b^2 = B^2+1$.

Solution 4.

From $(\sqrt{2}-1)^1 = \sqrt{2}-1$, $(\sqrt{2}-1)^2 = 3-2\sqrt{2}$, $(\sqrt{2}-1)^3 = 5\sqrt{2}-7$, $(\sqrt{2}-1)^4 = 17-12\sqrt{2}$, etc, we conjecture that

$$(*) \qquad (\sqrt{2}-1)^n = s_n\sqrt{2}+t_n$$

where $s_1 = 1$, $t_1 = 1$, $s_{n+1} = (-1)^n(|s_n|+|t_n|)$, $t_{n+1} = (-1)^{n+1}(2|s_n|+|t_n|)$.

Note that s_n is positive (negative) if n is odd (even) and t_n is negative (positive) if n is odd (even).

We now show by induction that (*) holds and that each $s_n\sqrt{2}+t_n$ of the form $\sqrt{m}-\sqrt{m-1}$ for some m.

It is easily verified that (*) is correct for $n = 1$ and 2. Assume (*) holds for some $n \geq 2$. Then

$$(\sqrt{2}-1)^{n+1} = (s_n\sqrt{2}+t_n)(\sqrt{2}-1) = (t_n - s_n)\sqrt{2} + (2s_n - t_n).$$

If n is odd, then

$$\begin{aligned} t_n - s_n &= -(|t_n|+|s_n|) = s_{n+1} \\ 2s_n - t_n &= 2|s_n|+|t_n| = t_{n+1}. \end{aligned}$$

If n is even, then

$$\begin{aligned} t_n - s_n &= |t_n|+|s_n| = s_{n+1} \\ 2s_n - t_n &= -2|s_n|-|t_n| = t_{n+1}. \end{aligned}$$

We have shown that (*) is correct for all n.

Observe now that $(s_{n+1}\sqrt{2})^2 - t_{n+1}^2 = 2(s_n^2 - 2s_nt_n + t_n^2) - (4s_n^2 - 4s_nt_n + t_n^2) = -2s_n^2 + t_n^2$ $= -((s_n\sqrt{2})^2 - t_n^2)$. Since $(s_1\sqrt{2})^2 - t_1^2 = 1$, it follows that $(s_n\sqrt{2})^2 - t_n^2 = (-1)^{n+1}$ for all n. To complete the proof it suffices to take $m = (s_n\sqrt{2})^2$, $m-1 = t_n^2$ when n is odd and $m = t_n^2$, $m-1 = (s_n\sqrt{2})^2$ when n is even.

QUESTION 3

First observe that if two neighbours have the same response on the n^{th} vote, then they both will respond the same way on the $(n+1)^{th}$ vote. Moreover, neither will ever change his response after the n^{th} vote.

Let A_n be the set of men who agree with at least one of their neighbours on the n^{th} vote. The previous paragraph says that $A_n \subset A_{n+1}$ for every $n \geq 1$. Moreover, we will be done if we can show that A_n contains all 25 men for some n.

Since there are an odd number of men at the table, it is not pssible that every man disagrees with both of his neighbours on the first vote. Therefore A_1 contains at least two men. And since $A_n \subset A_{n+1}$ for every n, there exists a $T < 25$ such that $A_T = A_{T+1}$. Suppose that A_T does not contain all 25 men; we shall use this to derive a contradiction. Since A_T is not empty, there must exist two neighbours, whom we shall call x and y, such that $x \in A_T$ and $y \notin A_T$. Since $x \in A_T$, he will respond the same way on the T^{th} and $(T+1)^{th}$ votes. But $y \notin A_T$, so y's response on the T^{th} vote differs from x's response. In fact, we know that y disagrees with both of his neighbours on the T^{th} vote, and so he will change his response on the $(T+1)^{th}$ vote. Therefore, on the $(T+1)^{th}$ vote, y responds the same way as does x. This implies that $y \in A_{T+1}$. But $y \notin A_T$, which contradicts the fact that $A_T = A_{T+1}$. Therefore we conclude that A_T contains all 25 men, and we are done.

QUESTION 4

There are three cases to be considered:

Case 1: If P is outside Ω (see figures I, II, and III), then since $\angle AUB = \angle AVB = \pi/2$, we have

$$\cos(\angle APB) = \frac{PU}{PB} = \frac{PV}{PA} = \sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = \sqrt{st}.$$

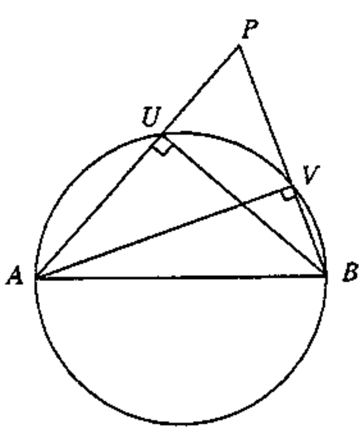

Figure I

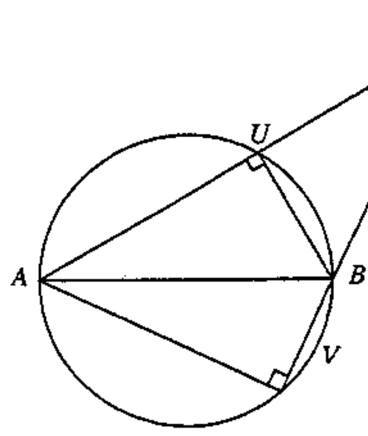

Figure II

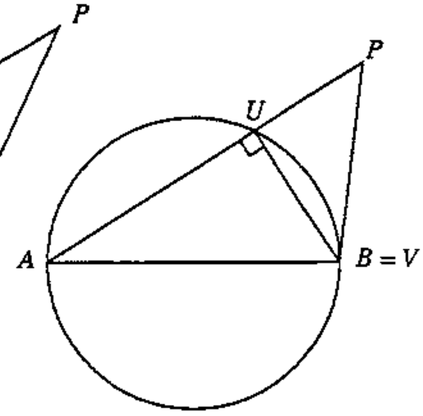

Figure III

Case 2: If P is on Ω (see figure IV), then

$$P = U = V \Rightarrow PU = PV = 0 \Rightarrow s = t = 0.$$

Since $\angle APB = \pi/2$, $\cos(\angle APB) = 0 = \sqrt{st}$ holds again.

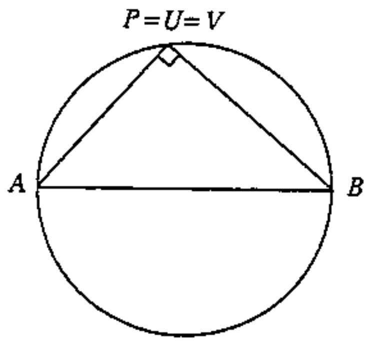

Figure IV

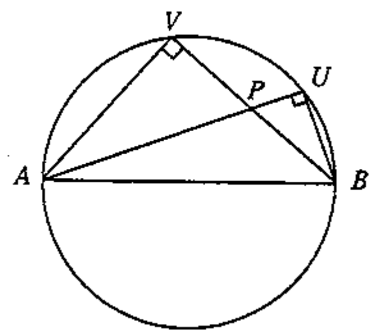

Figure V

Case 3: If P is inside Ω (figure V), then

$$\cos(\angle APB) = \cos(\pi - \angle APV) = -\cos(\angle APV) = -\frac{PV}{PA},$$

and

$$\cos(\angle APB) = \cos(\pi - \angle BPU) = -\cos(\angle BPU) = -\frac{PU}{PB}.$$

Therefore $\cos(\angle APB) = -\sqrt{\frac{PU}{PA} \cdot \frac{PV}{PB}} = -\sqrt{st}$.

QUESTION 5

100 degree angle question? bit harder?

Solution 1.

From A draw a lkine ℓ parallel to BC. Extend DF and DE to meet ℓ at P and Q respectively (See Figure I). Then from similar triangles, we have

$$\frac{AP}{BD} = \frac{AF}{FB} \text{ qne } \frac{AQ}{CD} = \frac{AE}{EC}$$

or

$$AP = \frac{AF}{FB} \cdot BD \text{ and } AQ = \frac{AE}{EC} \cdot CD. \qquad (1)$$

By Ceva's Theorem, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$ and thus

$$\frac{AF}{FB} \cdot BD = \frac{AE}{EC} \cdot CD \qquad (2)$$

From (1) and (2) we get $AP = AQ$ and hence $\triangle ADP \simeq \triangle ADQ$ from which $\angle EDH = \angle FDH$ follows.

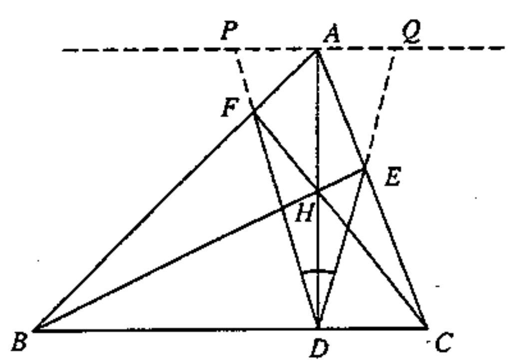

Figure I

Solution 2.

Use cartesian coordinates, with D at $(0,0)$, $A = (0,a)$, $B = (-b,0)$, $C = (c,0)$. Let $H = (0,h)$, $E = (u,v)$ and $F = (-r,s)$ where a,b,c,h,u,v,r,s are all positive (See Figure II).

It clearly suffices to show that $\frac{v}{u} = \frac{s}{r}$. Since EC and AC have the same slope, we have $\frac{v}{u-c} = \frac{a}{-c}$. Similarly, since EB and HB have the same slope, $\frac{v}{u+b} = \frac{k}{b}$. Thus

$$\frac{v}{a} = \frac{u-c}{-c} = \frac{-u}{c} + 1 \qquad (1)$$

and

$$\frac{v}{h} = \frac{u+b}{b} = \frac{u}{b} + 1 \qquad (2)$$

(2)-(1) we get $v(\frac{1}{h} - \frac{1}{a}) = u(\frac{1}{b} + \frac{1}{c})$ and thus

$$\frac{v}{u} = \frac{\frac{1}{b} + \frac{1}{c}}{\frac{1}{h} - \frac{1}{a}} = \frac{ah(b+c)}{bc(a-h)}.$$

With u, v, b and c replaced by $-r, s, -c$ and $-b$ respectively, we have, by a similar argument that

$$\frac{s}{-r} = \frac{ah(-c-b)}{bc(a-h)} \text{ or } \frac{s}{r} = \frac{ah(b+c)}{bc(a-h)}.$$

Therefore, $\frac{v}{u} = \frac{s}{r}$ as desired.

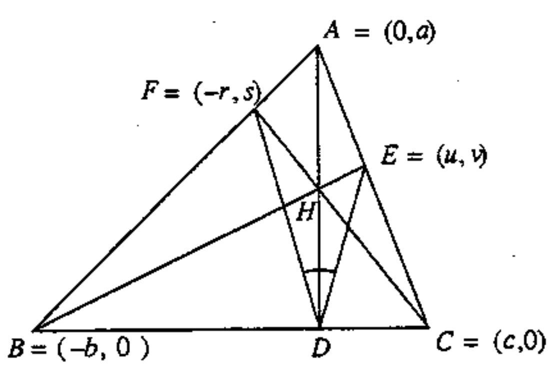

Figure II