Official Problem Set

1. Amy has drawn three points in a plane, $A$, $B$, and $C$, such that $AB = BC = CA = 6$. Amy is allowed to draw a new point if it is the circumcenter of a triangle whose vertices she has already drawn. For example, she can draw the circumcenter $O$ of triangle $ABC$, and then afterwards she can draw the circumcenter of triangle $ABO$.

(a) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 7.

(b) Prove that Amy can eventually draw a point whose distance from a previously drawn point is greater than 2019.

(Recall that the circumcenter of a triangle is the center of the circle that passes through its three vertices.)

2. Let $a$ and $b$ be positive integers such that $a + b^2$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

3. Let $m$ and $n$ be positive integers. A $2m \times 2n$ grid of squares is coloured in the usual chessboard fashion. Find the number of ways of placing $mn$ counters on the white squares, at most one counter per square, so that no two counters are on white squares that are diagonally adjacent. An example of a way to place the counters when $m = 2$ and $n = 3$ is shown below.
4. Let \( n \) be an integer greater than 1, and let \( a_0, a_1, \ldots, a_n \) be real numbers with \( a_1 = a_{n-1} = 0 \). Prove that for any real number \( k \),

\[
|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.
\]

5. David and Jacob are playing a game of connecting \( n \geq 3 \) points drawn in a plane. No three of the points are collinear. On each player’s turn, he chooses two points to connect by a new line segment. The first player to complete a cycle consisting of an odd number of line segments loses the game. (Both endpoints of each line segment in the cycle must be among the \( n \) given points, not points which arise later as intersections of segments.) Assuming David goes first, determine all \( n \) for which he has a winning strategy.

**Important!**

*Please do not discuss this problem set online for at least 24 hours.*