1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\frac{a_1}{1 + a_1} + \frac{a_2}{(1 + a_1)(1 + a_2)} + \frac{a_3}{(1 + a_1)(1 + a_2)(1 + a_3)} + \cdots + \frac{a_n}{(1 + a_1)(1 + a_2) \cdots (1 + a_n)}$$

is greater than or equal to $2^n - 1$.

2. Let $m$ and $n$ be odd positive integers. Each square of an $m$ by $n$ board is coloured red or blue. A row is said to be red-dominated if there are more red squares than blue squares in the row. A column is said to be blue-dominated if there are more blue squares than red squares in the column. Determine the maximum possible value of the number of red-dominated rows plus the number of blue-dominated columns. Express your answer in terms of $m$ and $n$.

3. Let $p$ be a fixed odd prime. A $p$-tuple $(a_1, a_2, a_3, \ldots, a_p)$ of integers is said to be good if

   (i) $0 \leq a_i \leq p - 1$ for all $i$, and
   (ii) $a_1 + a_2 + a_3 + \cdots + a_p$ is not divisible by $p$, and
   (iii) $a_1a_2 + a_2a_3 + a_3a_4 + \cdots + a_pa_1$ is divisible by $p$.

Determine the number of good $p$-tuples.

4. The quadrilateral $ABCD$ is inscribed in a circle. The point $P$ lies in the interior of $ABCD$, and $\angle PAB = \angle PBC = \angle PCD = \angle PDA$. The lines $AD$ and $BC$ meet at $Q$, and the lines $AB$ and $CD$ meet at $R$. Prove that the lines $PQ$ and $PR$ form the same angle as the diagonals of $ABCD$.

5. Fix positive integers $n$ and $k \geq 2$. A list of $n$ integers is written in a row on a blackboard. You can choose a contiguous block of integers, and I will either add 1 to all of them or subtract 1 from all of them. You can repeat this step as often as you like, possibly adapting your selections based on what I do. Prove that after a finite number of steps, you can reach a state where at least $n - k + 2$ of the numbers on the blackboard are all simultaneously divisible by $k$. 