Sun Life Financial

45^{th} Canadian Mathematical Olympiad

Wednesday, March 27, 2013

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1. Determine all polynomials $P(x)$ with real coefficients such that

$$(x+1)P(x-1)-(x-1)P(x)$$

is a constant polynomial.

2. The sequence $a_1, a_2, \ldots, a_n$ consists of the numbers $1, 2, \ldots, n$ in some order. For which positive integers n is it possible that the $n+1$ numbers 0, a_1, a_1+a_2, $a_1+a_2+a_3$, $\ldots$, $a_1+a_2+\cdots+a_n$ all have different remainders when divided by $n+1$?

3. Let G be the centroid of a right-angled triangle ABC with $\angle BCA = 90^\circ$. Let P be the point on ray AG such that $\angle CPA = \angle CAB$, and let Q be the point on ray BG such that $\angle CQB = \angle ABC$. Prove that the circumcircles of triangles AQG and BPG meet at a point on side AB.

4. Let n be a positive integer. For any positive integer j and positive real number r, define $f_j(r)$ and $g_j(r)$ by

$$f_j(r) = \min(jr, n) + \min\left(\frac{j}{r}, n\right), \quad \text{and} \quad g_j(r) = \min(\lceil jr \rceil, n) + \min\left(\left\lceil \frac{j}{r} \right\rceil, n\right),$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Prove that

$$\sum_{j=1}^{n} f_j(r) \le n^2 + n \le \sum_{j=1}^{n} g_j(r)$$

for all positive real numbers r.

5. Let O denote the circumcentre of an acute-angled triangle ABC. Let point P on side AB be such that $\angle BOP = \angle ABC$, and let point Q on side AC be such that $\angle COQ = \angle ACB$. Prove that the reflection of BC in the line PQ is tangent to the circumcircle of triangle APQ.