38th Canadian Mathematical Olympiad

Wednesday, March 29,2006

1. Let $f(n,k)$ be the number of ways of distributing k candies to n children so that each child receives at most 2 candies. For example, if $n=3$, then $f(3,7)=0$, $f(3,6)=1$ and $f(3,4)=6$.

   Determine the value of

$$f(2006,1)+f(2006,4)+f(2006,7)+\cdots+f(2006,1000)+f(2006,1003)\ .$$

2. Let ABC be an acute-angled triangle. Inscribe a rectangle $DEFG$ in this triangle so that D is on AB, E is on AC and both F and G are on BC. Describe the locus of (*i.e.*, the curve occupied by) the intersections of the diagonals of all possible rectangles $DEFG$.

3. In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m=n$.

4. Consider a round-robin tournament with $2n+1$ teams, where each team plays each other team exactly once. We say that three teams X, Y and Z, form a *cycle triplet* if X beats Y, Y beats Z, and Z beats X. There are no ties.

   (a) Determine the minimum number of cycle triplets possible.

   (b) Determine the maximum number of cycle triplets possible.

5. The vertices of a right triangle ABC inscribed in a circle divide the circumference into three arcs. The right angle is at A, so that the opposite arc BC is a semicircle while arc AB and arc AC are supplementary. To each of the three arcs, we draw a tangent such that its point of tangency is the midpoint of that portion of the tangent intercepted by the extended lines AB and AC. More precisely, the point D on arc BC is the midpoint of the segment joining the points D' and D'' where the tangent at D intersects the extended lines AB and AC. Similarly for E on arc AC and F on arc AB.

   Prove that triangle DEF is equilateral.