1. Determine the number of real solutions $a$ to the equation

$$\left\lfloor \frac{1}{2} a \right\rfloor + \left\lfloor \frac{1}{3} a \right\rfloor + \left\lfloor \frac{1}{5} a \right\rfloor = a.$$ 

Here, if $x$ is a real number, then $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to $x$.

2. Find all real numbers $x$ such that

$$x = \left( x - \frac{1}{x} \right)^{1/2} + \left( 1 - \frac{1}{x} \right)^{1/2}.$$ 

3. Let $n$ be a natural number such that $n \geq 2$. Show that

$$\frac{1}{n+1} \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right) > \frac{1}{n} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} \right).$$

4. Let $ABC$ be a triangle with $\angle BAC = 40^\circ$ and $\angle ABC = 60^\circ$. Let $D$ and $E$ be the points lying on the sides $AC$ and $AB$, respectively, such that $\angle CBD = 40^\circ$ and $\angle BCE = 70^\circ$. Let $F$ be the point of intersection of the lines $BD$ and $CE$. Show that the line $AF$ is perpendicular to the line $BC$.

5. Let $m$ be a positive integer. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_0 = 0$, $a_1 = m$, and $a_{n+1} = m^2a_n - a_{n-1}$ for $n = 1, 2, 3, \ldots$. Prove that an ordered pair $(a, b)$ of non-negative integers, with $a \leq b$, gives a solution to the equation

$$\frac{a^2 + b^2}{ab + 1} = m^2$$

if and only if $(a, b)$ is of the form $(a_n, a_{n+1})$ for some $n \geq 0$. 