Canadian Mathematical Olympiad
1995

PROBLEM 1

Let $f(x) = \frac{9^x}{9^x+3}$. Evaluate the sum

$$f(\frac{1}{1996}) + f(\frac{2}{1996}) + f(\frac{3}{1996}) + \cdots + f(\frac{1995}{1996})$$

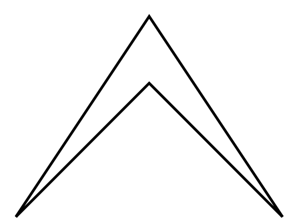
PROBLEM 2

Let a, b, and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

PROBLEM 3

Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees. (See Figure displayed.) Let C be a convex polygon having 5 sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + \frac{s-2}{2}$.



PROBLEM 4

Let n be a fixed positive integer. Show that for only nonnegative integers k, the diophantine equation

$$x_1^3 + x_2^3 + \cdots + x_n^3 = y^{3k+2}$$

has infinitely many solutions in positive integers x_i and y.

PROBLEM 5

Suppose that u is a real parameter with $0 < u < 1$. Define

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq u \\ 1 - \left(\sqrt{ux} + \sqrt{(1-u)(1-x)}\right)^2 & \text{if } u \leq x \leq 1 \end{cases}$$

and define the sequence $\{u_n\}$ recursively as follows:

$$u_1 = f(1), \text{ and } u_n = f(u_{n-1}) \text{ for all } n > 1.$$

Show that there exists a positive ineger k for which $u_k = 0$.