Solutions of APMO 2013

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Solution. Let M and N be midpoints of sides BC and AC, respectively. Notice that $\angle MOC = \frac{1}{2}\angle BOC = \angle EAB$, $\angle OMC = 90^\circ = \angle AEB$, so triangles OMC and AEB are similar and we get $\frac{OM}{AE} = \frac{OC}{AB}$. For triangles ONA and BDA we also have $\frac{ON}{BD} = \frac{OA}{BA}$. Then $\frac{OM}{AE} = \frac{ON}{BD}$ or $BD \cdot OM = AE \cdot ON$.

Denote by $S(\Phi)$ the area of the figure Φ. So, we see that $S(OBD) = \frac{1}{2}BD \cdot OM = \frac{1}{2}AE \cdot ON = S(OAE)$. Analogously, $S(OCD) = S(OAF)$ and $S(OCE) = S(OBF)$.

Alternative solution. Let R be the circumradius of triangle ABC, and as usual write A, B, C for angles $\angle CAB$, $\angle ABC$, $\angle BCA$ respectively, and a, b, c for sides BC, CA, AB respectively. Then the area of triangle OCD is

$$S(OCD) = \tfrac{1}{2} \cdot OC \cdot CD \cdot \sin(\angle OCD) = \tfrac{1}{2}R \cdot CD \cdot \sin(\angle OCD).$$

Now $CD = b\cos C$, and

$$\angle OCD = \frac{180^\circ - 2A}{2} = 90^\circ - A$$

(since triangle OBC is isosceles, and $\angle BOC = 2A$). So

$$S(OCD) = \tfrac{1}{2}Rb\cos C\sin(90^\circ - A) = \tfrac{1}{2}Rb\cos C\cos A.$$

A similar calculation gives

$$\begin{aligned} S(OAF) &= \tfrac{1}{2}OA \cdot AF \cdot \sin(\angle OAF) \\ &= \tfrac{1}{2}R \cdot (b\cos A)\sin(90^\circ - C) \\ &= \tfrac{1}{2}Rb\cos A\cos C, \end{aligned}$$

so OCD and OAF have the same area. In the same way we find that OBD and OAE have the same area, as do OCE and OBF.

Problem 2. Determine all positive integers n for which $\frac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Solution. We will show that there are no positive integers n satisfying the condition of the problem.

Let $m = [\sqrt{n}]$ and $a = n - m^2$. We have $m \geq 1$ since $n \geq 1$. From $n^2+1 = (m^2+a)^2+1 \equiv (a-2)^2+1 \pmod{(m^2+2)}$, it follows that the condition of the problem is equivalent to the fact that $(a-2)^2+1$ is divisible by m^2+2. Since we have

$$0 < (a-2)^2 + 1 \leq \max\{2^2, (2m-2)^2\} + 1 \leq 4m^2 + 1 < 4(m^2+2),$$

we see that $(a-2)^2+1 = k(m^2+2)$ must hold with $k = 1, 2$ or 3. We will show that none of these can occur.

Case 1. When $k = 1$. We get $(a-2)^2 - m^2 = 1$, and this implies that $a-2 = \pm 1$, $m = 0$ must hold, but this contradicts with fact $m \geq 1$.

Case 2. When $k = 2$. We have $(a-2)^2+1 = 2(m^2+2)$ in this case, but any perfect square is congruent to $0, 1, 4$ mod 8, and therefore, we have $(a-2)^2+1 \equiv 1, 2, 5 \pmod 8$, while $2(m^2+2) \equiv 4, 6 \pmod 8$. Thus, this case cannot occur either.

Case 3. When $k = 3$. We have $(a-2)^2+1 = 3(m^2+2)$ in this case. Since any perfect square is congruent to 0 or 1 mod 3, we have $(a-2)^2+1 \equiv 1, 2 \pmod 3$, while $3(m^2+2) \equiv 0 \pmod 3$, which shows that this case cannot occur either.

Problem 3. For $2k$ real numbers $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$ define the sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, \ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Solution. Let us write $A = \sum_{i=1}^{k} a_i$ and $B = \sum_{i=1}^{k} b_i$. Summing the corresponding terms of the following inequalities over i,

$$a_i n + b_i - 1 < [a_i n + b_i] \leq a_i n + b_i,$$

we obtain $An + B - k < X_n < An + B$. Now suppose that $\{X_n\}$ is an arithmetic progression with the common difference d, then we have $nd = X_{n+1} - X_1$ and $A + B - k < X_1 \leq A + B$ Combining with the inequalities obtained above, we get

$$A(n+1) + B - k < nd + X_1 < A(n+1) + B,$$

or

$$An - k \leq An + (A + B - X_1) - k < nd < An + (A + B - X_1) < An + k,$$

from which we conclude that $|A - d| < \frac{k}{n}$ must hold. Since this inequality holds for any positive integer n, we must have $A = d$. Since $\{X_n\}$ is a sequence of integers, d must be an integer also, and thus we conclude that A is also an integer.

Problem 4. Let a and b be positive integers, and let A and B be finite sets of integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then $i + a$ belongs to A or $i - b$ belongs to B.

Prove that $a|A| = b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Solution. Let $A^* = \{n - a : n \in A\}$ and $B^* = \{n + b : n \in B\}$. Then, by (ii), $A \cup B \subseteq A^* \cup B^*$ and by (i),

$$|A \cup B| \leq |A^* \cup B^*| \leq |A^*| + |B^*| = |A| + |B| = |A \cup B|. \qquad (1)$$

Thus, $A \cup B = A^* \cup B^*$ and A^* and B^* have no element in common. For each finite set X of integers, let $\sum(X) = \sum_{x \in X} x$. Then

$$\begin{aligned} \sum(A) + \sum(B) &= \sum(A \cup B) \\ &= \sum(A^* \cup B^*) = \sum(A^*) + \sum(B^*) \\ &= \sum(A) - a|A| + \sum(B) + b|B|, \end{aligned} \tag{2}$$

which implies $a|A| = b|B|$.

Alternative solution. Let us construct a directed graph whose vertices are labelled by the members of $A \cup B$ and such that there is an edge from i to j iff $j \in A$ and $j = i + a$ or $j \in B$ and $j = i - b$. From (ii), each vertex has out-degree ≥ 1 and, from (i), each vertex has in-degree ≤ 1. Since the sum of the out-degrees equals the sum of the in-degrees, each vertex has in-degree and out-degree equal to 1. This is only possible if the graph is the union of disjoint cycles, say $G_1, G_2, \ldots, G_n$. Let $|A_k|$ be the number of elements of A in G_k and $|B_k|$ be the number of elements of B in G_k. The cycle G_k will involve increasing vertex labels by a a total of $|A_k|$ times and decreasing them by b a total of $|B_k|$ times. Since it is a cycle, we have $a|A_k| = b|B_k|$. Summing over all cycles gives the result.

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Solution. To show B, E, R are collinear, it is equivalent to show the lines AD, BE, CQ are concurrent. Let CQ intersect AD at R and BE intersect AD at R'. We shall show $RD/RA = R'D/R'A$ so that $R = R'$.

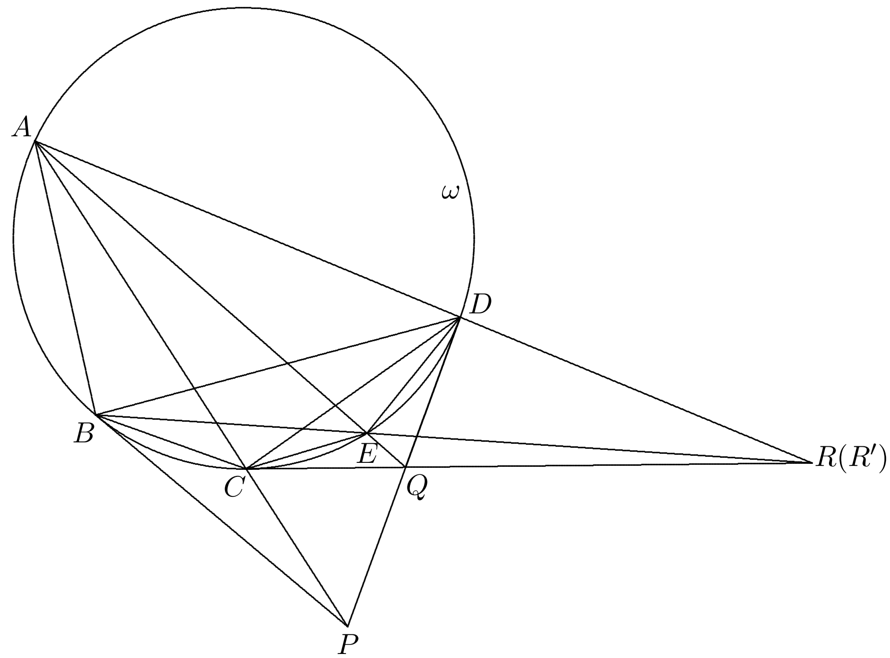
Since $\triangle PAD$ is similar to $\triangle PDC$ and $\triangle PAB$ is similar to $\triangle PBC$, we have $AD/DC = PA/PD = PA/PB = AB/BC$. Hence, $AB \cdot DC = BC \cdot AD$. By Ptolemy's theorem, $AB \cdot DC = BC \cdot AD = \frac{1}{2} CA \cdot DB$. Similarly $CA \cdot ED = CE \cdot AD = \frac{1}{2} AE \cdot DC$.

Thus

$$\frac{DB}{AB} = \frac{2DC}{CA}, \tag{3}$$

and

$$\frac{DC}{CA} = \frac{2ED}{AE}. \tag{4}$$



Since the triangles RDC and RCA are similar, we have $\frac{RD}{RC} = \frac{DC}{CA} = \frac{RC}{RA}$. Thus using (4)

$$\frac{RD}{RA} = \frac{RD \cdot RA}{RA^2} = \left(\frac{RC}{RA}\right)^2 = \left(\frac{DC}{CA}\right)^2 = \left(\frac{2ED}{AE}\right)^2. \tag{5}$$

Using the similar triangles ABR' and EDR', we have $R'D/R'B = ED/AB$. Using the similar triangles DBR' and EAR' we have $R'A/R'B = EA/DB$. Thus using (3) and (4),

$$\frac{R'D}{R'A} = \frac{ED \cdot DB}{EA \cdot AB} = \left(\frac{2ED}{AE}\right)^2. \tag{6}$$

It follows from (5) and (6) that $R = R'$.