XVII APMO - March, 2005

Problems and Solutions

Problem 1. Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

(Solution) Let a be an irrational number. If a^2 is irrational, we let $b=-a$. Then, $a+b=0$ is rational and $ab=-a^2$ is irrational. If a^2 is rational, we let $b=a^2-a$. Then, $a+b=a^2$ is rational and $ab=a^2(a-1)$. Since

$$a=\frac{ab}{a^2}+1$$

is irrational, so is ab.

Now, we let $b'=\frac{1}{a}$ or $b'=\frac{2}{a}$. Then $ab'=1$ or 2, which is rational. Note that

$$a+b'=\frac{a^2+1}{a} \quad \text{or} \quad a+b'=\frac{a^2+2}{a}.$$

Since,

$$\frac{a^2+2}{a}-\frac{a^2+1}{a}=\frac{1}{a},$$

at least one of them is irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \;\geq\; \frac{4}{3}\,.$$

(Solution) Observe that

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}\,. \tag{1}$$

In fact, this is equivalent to $(2+x^2)^2 \geq 4(1+x^3)$, or $x^2(x-2)^2 \geq 0$. Notice that equality holds in (1) if and only if $x = 2$.

We substitute x by a, b, c in (1), respectively, to find

$$\begin{aligned} &\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \\ &\geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+a^2)}\,. \end{aligned} \tag{2}$$

We combine the terms on the right hand side of (2) to obtain

$$\text{Left hand side of (2)} \geq \frac{2S(a,b,c)}{36+S(a,b,c)} = \frac{2}{1+36/S(a,b,c)}\,, \tag{3}$$

where $S(a,b,c) := 2(a^2+b^2+c^2) + (ab)^2 + (bc)^2 + (ca)^2$. By AM-GM inequality, we have

$$\begin{aligned} a^2+b^2+c^2 &\geq 3\sqrt[3]{(abc)^2} = 12\,, \\ (ab)^2+(bc)^2+(ca)^2 &\geq 3\sqrt[3]{(abc)^4} = 48\,. \end{aligned}$$

Note that the equalities holds if and only if $a = b = c = 2$. The above inequalities yield

$$S(a,b,c) = 2(a^2+b^2+c^2) + (ab)^2 + (bc)^2 + (ca)^2 \geq 72\,. \tag{4}$$

Therefore

$$\frac{2}{1+36/S(a,b,c)} \geq \frac{2}{1+36/72} = \frac{4}{3}\,, \tag{5}$$

which is the required inequality.

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

(Solution) Suppose that one side of a triangle has length n. Then it can be cut into n^2 congruent triangles which are similar to the original one and whose corresponding sides to the side of length n have lengths 1.

Since $2005 = 5 \times 401$ where 5 and 401 are primes and both primes are of the type $4k+1$, it is representable as a sum of two integer squares. Indeed, it is easy to see that

$$\begin{aligned} 2005 &= 5 \times 401 = (2^2+1)(20^2+1) \\ &= 40^2 + 20^2 + 2^2 + 1 \\ &= (40-1)^2 + 2 \times 40 + 20^2 + 2^2 \\ &= 39^2 + 22^2. \end{aligned}$$

Let ABC be a right-angled triangle with the legs AB and BC having lengths 39 and 22, respectively. We draw the altitude BK, which divides ABC into two similar triangles. Now we divide ABK into 39^2 congruent triangles as described above and BCK into 22^2 congruent triangles. Since ABK is similar to BKC, all 2005 triangles will be congruent.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \le i, j \le n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \le \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i-k| + |j-\ell| = 1$.

(Solution) At most $n^2 + c^2 - nc - c$ houses can be saved. This can be achieved under the following order of defending:

$$\begin{array}{c}(2, c), (2, c+1); (3, c-1), (3, c+2); (4, c-2), (4, c+3); \dots \\ (c+1, 1), (c+1, 2c); (c+1, 2c+1), \dots, (c+1, n).\end{array} \tag{6}$$

Under this strategy, there are

2 columns (column numbers $c, c+1$) at which $n-1$ houses are saved
2 columns (column numbers $c-1, c+2$) at which $n-2$ houses are saved
...
2 columns (column numbers $1, 2c$) at which $n-c$ houses are saved
$n-2c$ columns (column numbers $n-2c+1, \dots, n$) at which $n-c$ houses are saved

Adding all these we obtain:

$$2[(n-1) + (n-2) + \cdots + (n-c)] + (n-2c)(n-c) = n^2 + c^2 - cn - c. \tag{7}$$

We say that a house indexed by (i, j) is at level t if $|i-1| + |j-c| = t$. Let $d(t)$ be the number of houses at level t defended by time t, and $p(t)$ be the number of houses at levels greater than t defended by time t. It is clear that

$$p(t) + \sum_{i=1}^{t} d(i) \le t \text{ and } p(t+1) + d(t+1) \le p(t) + 1.$$

Let $s(t)$ be the number of houses at level t which are not burning at time t. We prove that

$$s(t) \le t - p(t) \le t$$

for $1 \le t \le n-1$ by induction. It is obvious when $t = 1$. Assume that it is true for $t = k$. The union of the neighbors of any $k - p(k) + 1$ houses at level $k+1$ contains at least $k - p(k) + 1$ vertices at level k. Since $s(k) \le k - p(k)$, one of these houses at level k is burning. Therefore, at most $k - p(k)$ houses at level $k+1$ have no neighbor burning. Hence we have

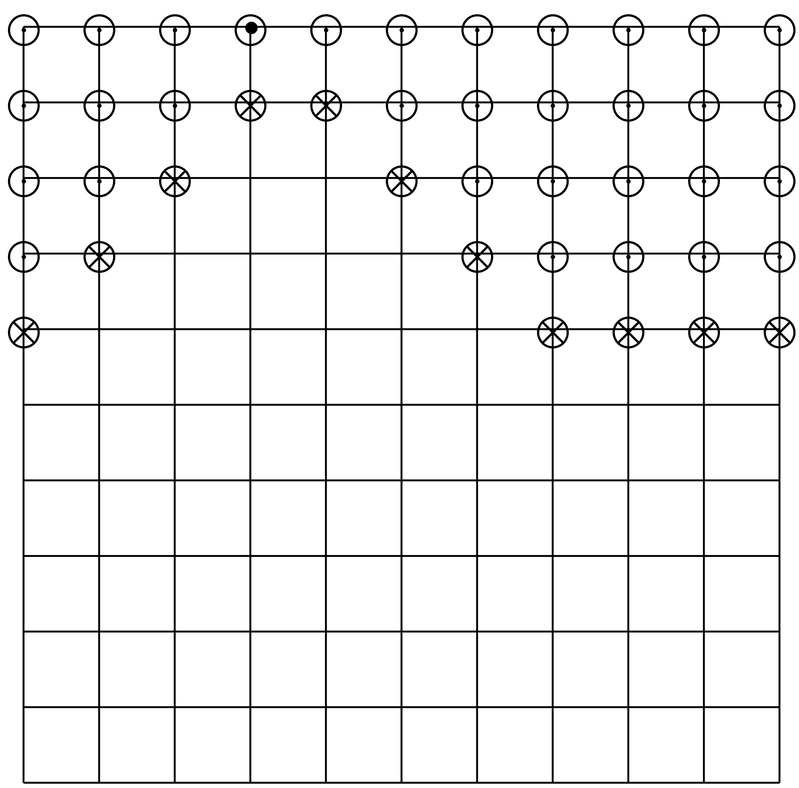
$$\begin{aligned} s(k+1) &\le k - p(k) + d(k+1) \\ &= (k+1) - (p(k) + 1 - d(k+1)) \\ &\le (k+1) - p(k+1). \end{aligned}$$

We now prove that the strategy given above is optimal. Since

$$\sum_{t=1}^{n-1} s(t) \leq \binom{n}{2},$$

the maximum number of houses at levels less than or equal to $n-1$, that can be saved under any strategy is at most $\binom{n}{2}$, which is realized by the strategy above. Moreover, at levels bigger than $n-1$, every house is saved under the strategy above.

The following is an example when $n = 11$ and $c = 4$. The houses with $\bigcirc$ mark are burned. The houses with $\otimes$ mark are blocked ones and hence those and the houses below them are saved.



Problem 5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

(Solution) Let ω, O and I be the circumcircle, the circumcenter and the incenter of ABC, respectively. Let D be the point of intersection of the line BI and the circle ω such that $D \neq B$. Then D is the midpoint of the arc AC. Hence $OD \perp CN$ and $OD = R$.

We first show that triangles MNC and IOD are similar. Because $BC = BM$, the line BI (the bisector of $\angle MBC$) is perpendicular to the line CM. Because $OD \perp CN$ and $ID \perp MC$, it follows that

$$\angle ODI = \angle NCM \tag{8}$$

Let $\angle ABC = 2\beta$. In the triangle BCM, we have

$$\frac{CM}{NC} = \frac{CM}{BC} = 2\sin\beta \tag{9}$$

Since $\angle DIC = \angle DCI$, we have $ID = CD = AD$. Let E be the point of intersection of the line DO and the circle ω such that $E \neq D$. Then DE is a diameter of ω and $\angle DEC = \angle DBC = \beta$. Thus we have

$$\frac{DI}{OD} = \frac{CD}{OD} = \frac{2R\sin\beta}{R} = 2\sin\beta. \tag{10}$$

Combining equations (8), (9), and (10) shows that triangles MNC and IOD are similar. It follows that

$$\frac{MN}{BC} = \frac{MN}{NC} = \frac{IO}{OD} = \frac{IO}{R}\,. \tag{11}$$

The well-known Euler's formula states that

$$OI^2 = R^2 - 2Rr. \tag{12}$$

Therefore,

$$\frac{MN}{BC} = \sqrt{1 - \frac{2r}{R}}\,. \tag{13}$$

(Alternative Solution) Let a (resp., b, c) be the length of BC (resp., AC, AB). Let α (resp., β, γ) denote the angle $\angle BAC$ (resp., $\angle ABC$, $\angle ACB$). By introducing coordinates $B = (0,0)$, $C = (a,0)$, it is immediate that the coordinates of M and N are

$$M = (a\cos\beta, a\sin\beta), \quad N = (a - a\cos\gamma, a\sin\gamma), \tag{14}$$

respectively. Therefore,

$$
\begin{aligned}
(MN/BC)^2 &= [(a - a\cos\gamma - a\cos\beta)^2 + (a\sin\gamma - a\sin\beta)^2]/a^2 \\
&= (1 - \cos\gamma - \cos\beta)^2 + (\sin\gamma - \sin\beta)^2 \\
&= 3 - 2\cos\gamma - 2\cos\beta + 2(\cos\gamma\cos\beta - \sin\gamma\sin\beta) \\
&= 3 - 2\cos\gamma - 2\cos\beta + 2\cos(\gamma + \beta) \\
&= 3 - 2\cos\gamma - 2\cos\beta - 2\cos\alpha \\
&= 3 - 2(\cos\gamma + \cos\beta + \cos\alpha).
\end{aligned} \tag{15}
$$

Now we claim

$$
\cos\gamma + \cos\beta + \cos\alpha = \frac{r}{R} + 1. \tag{16}
$$

From

$$
\begin{aligned}
a &= b\cos\gamma + c\cos\beta \\
b &= c\cos\alpha + a\cos\gamma \\
c &= a\cos\beta + b\cos\alpha
\end{aligned} \tag{17}
$$

we get

$$
a(1 + \cos\alpha) + b(1 + \cos\beta) + c(1 + \cos\gamma) = (a + b + c)(\cos\alpha + \cos\beta + \cos\gamma). \tag{18}
$$

Thus

$$
\begin{aligned}
&\cos\alpha + \cos\beta + \cos\gamma \\
&= \frac{1}{a+b+c}\left(a(1 + \cos\alpha) + b(1 + \cos\beta) + c(1 + \cos\gamma)\right) \\
&= \frac{1}{a+b+c}\left(a\left(1 + \frac{b^2 + c^2 - a^2}{2bc}\right) + b\left(1 + \frac{a^2 + c^2 - b^2}{2ac}\right) + c\left(1 + \frac{a^2 + b^2 - c^2}{2ab}\right)\right) \\
&= \frac{1}{a+b+c}\left(a + b + c + \frac{a^2(b^2 + c^2 - a^2) + b^2(a^2 + c^2 - b^2) + c^2(a^2 + b^2 - c^2)}{2abc}\right) \\
&= 1 + \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2abc(a+b+c)}.
\end{aligned} \tag{19}
$$

On the other hand, from $R = \dfrac{a}{2\sin\alpha}$ it follows that

$$
\begin{aligned}
R^2 &= \frac{a^2}{4(1 - \cos^2\alpha)} = \frac{a^2}{4\left(1 - \left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)^2\right)} \\
&= \frac{a^2b^2c^2}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}.
\end{aligned} \tag{20}
$$

Also from $\frac{1}{2}(a+b+c)r = \frac{1}{2}bc\sin\alpha$, it follows that

$$\begin{aligned} r^2 &= \frac{b^2c^2(1-\cos^2\alpha)}{(a+b+c)^2} = \frac{b^2c^2\left(1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right)}{(a+b+c)^2} \\ &= \frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4(a+b+c)^2}. \end{aligned} \tag{21}$$

Combining (19), (20) and (21), we get (16) as desired.

Finally, by (15) and (16) we have

$$\frac{MN}{BC} = \sqrt{1-\frac{2r}{R}}. \tag{22}$$

Another proof of (16) from R.A. Johnson's "Advanced Euclidean Geometry"[1]:

Construct the perpendicular bisectors OD, OE, OF, where D, E, F are the midpoints of BC, CA, AB, respectively. By Ptolemy's Theorem applied to the cyclic quadrilateral $OEAF$, we get

$$\frac{a}{2}\cdot R = \frac{b}{2}\cdot OF + \frac{c}{2}\cdot OE.$$

Similarly

$$\frac{b}{2}\cdot R = \frac{c}{2}\cdot OD + \frac{a}{2}\cdot OF, \quad \frac{c}{2}\cdot R = \frac{a}{2}\cdot OE + \frac{b}{2}\cdot OD.$$

Adding, we get

$$sR = OD\cdot\frac{b+c}{2} + OE\cdot\frac{c+a}{2} + OF\cdot\frac{a+b}{2}, \tag{23}$$

where s is the semiperimeter. But also, the area of triangle OBC is $OD\cdot\frac{a}{2}$, and adding similar formulas for the areas of triangles OCA and OAB gives

$$rs = \triangle ABC = OD\cdot\frac{a}{2} + OE\cdot\frac{b}{2} + OF\cdot\frac{c}{2} \tag{24}$$

Adding (23) and (24) gives $s(R+r) = s(OD+OE+OF)$, or

$$OD+OE+OF = R+r.$$

Since $OD = R\cos A$ *etc.*, (16) follows.

[1]This proof was introduced to the coordinating country by Professor Bill Sands of Canada.