THE 1993 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral such that all sides have equal length and angle ABC is 60 deg. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF.
Prove that $CA^2 = CM \times CE$.

Question 2

Find the total number of different integer values the function

$$f(x) = [x] + [2x] + [\frac{5x}{3}] + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

Question 3

Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{and} \\ g(x) &= c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x + r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Question 4

Determine all positive integers n for which the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

has an integer as a solution.

Question 5

Let P_1, P_2, $\ldots$, $P_{1993} = P_0$ be distinct points in the xy-plane with the following properties:

(i) both coordinates of P_i are integers, for $i = 1, 2, \ldots, 1993$;

(ii) there is no point other than P_i and P_{i+1} on the line segment joining P_i with P_{i+1} whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.

Prove that for some i, $0 \leq i \leq 1992$, there exists a point Q with coordinates (q_x, q_y) on the line segment joining P_i with P_{i+1} such that both $2q_x$ and $2q_y$ are odd integers.